People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
TAHRI Mohamed University of Bechar
Faculty of Technology

COURSE MATERIAL

Fluid Mechanics

Courses and Solved Exercises

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2nd Year Bachelor's (Licence) in Civil Engineering

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Foreword

This course material, titled "Fluid Mechanics," is intended for second-year students in Civil Engineering and Public Works Engineering at Tahri Mohamed University of Bechar. The content of this course material aligns with the Civil Engineering Bachelor's degree curriculum taught in the Civil Engineering Department at Tahri Mohamed University.

Basic knowledge of fluid mechanics is presented in this course to better understand and assimilate the content of this educational material. Fluid mechanics (hydraulics) is a branch of applied mechanics that deals with the behavior of fluids at rest and in motion. In hydrostatics, specific weight is the most important property, while in hydrodynamics, density and viscosity are the dominant properties.

The purpose of this educational material is to introduce students to techniques for solving partial differential equations related to the general equation of motion using Euler variables. Euler's equations are used in this course to establish the general equation of motion and the general equation of equilibrium for fluids at rest.

This educational document is divided into four chapters:

- The first chapter is devoted to hydrostatics
- The second to the dynamics of perfect fluids
- The third chapter focuses on the dynamics of real fluids
- The fourth covers the calculation of water distribution networks

At the end of the course material, a bibliography is presented, including references to the documents used in the preparation of this course material.

Table of Contents

Chapter 1: Fluid properties

Chapter Objectives

By The End Of This Chapter, Students Should Be Able To:

- ✓ Distinguish Between The Different Categories Of Fluids;
- ✓ Identify And Describe The Main Physical Properties Of Fluids.

1 What is “Fluid Mechanics”

Mechanics is the field of science focused on the motion of material bodies. Mechanics involves force, energy, motion, deformation, and material properties. When mechanics applies to material bodies in the solid phase, the discipline is called solid mechanics. When the material body is in the gas or liquid phase, the discipline is called fluid mechanics. In summary, fluid mechanics is the science of energy, motion, deformation, and properties when the material is in the gas or liquid phase.

Fluid mechanics, a special branch of general mechanics, describes the laws of liquid and gas motion. Flows of liquids and gases play an important role in nature and in technical applications, as, for example, flows in living organisms, atmospheric circulation, oceanic currents, tidal flows in rivers, wind- and water loads on buildings and structures, gas motion in flames and explosions, aero- and hydrodynamic forces acting on airplanes and ships, flows in water and gas turbines, pumps, engines, pipes, valves, bearings, hydraulic systems, and others.

Fluid mechanics and hydraulics represent that branch of applied mechanics that deals with the behavior of fluids at rest and in motion. In the development of the principles of fluid mechanics, some fluid properties play principal roles, others only minor roles or no roles at all. In fluid statics, specific weight (or unit weight) is the important property, whereas in fluid flow, density and viscosity are predominant properties. Where appreciable compressibility occurs, principles of thermodynamics must be considered. Vapor pressure becomes important when negative pressures (gage) are involved, and surface tension affects static and flow conditions in small passages.

2 What is a fluid?

2.1 The four main states

In physics, a state of matter is one of the distinct forms in which matter can exist. Four states of matter are observable in everyday life: solid, liquid, gas, and plasma. Many intermediate states are known to exist, such as liquid crystal, and some states only exist under extreme conditions,

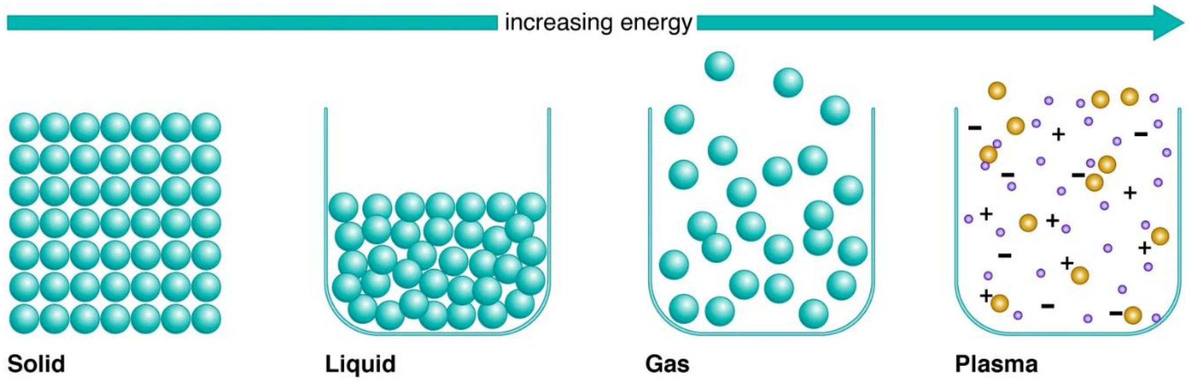


Figure 2-1 : Organisation of atoms or molecules of a material in its various states.

Table 2-1: some characteristics of the matter in its four possible states.

Solid	Fluid	Gaz	Plasma
Rigid	Not Rigid	Not Rigid	Not Rigid
Fixed Shape	No Fixed Shape	No Fixed Shape	No Fixed Shape
Fixed Volume	Fixed Volume	No Fixed Volume	No Fixed Volume
High Density	Average to High Density	Low Density	
Closely tight and	Closely tight but not disorganized particles	Far apart and disorganized particles	
organized particles	Slightly Compressible	Highly Compressible	
Slightly Compressible	Free surface		
Strong bond	Weak bond	No bond	Ionization
Regular arrangement	Irregular arrangement	Irregular arrangement	irregular arrangement
Vibrates almost in fixed position	Moves around each other	Moves in all directions	Moves in all directions

Figure 2-2 illustrates the four fundamental states of matter — Solid, Liquid, Gas, and Plasma — and the phase transitions between them. Moving from left to right requires the addition of thermal energy, while the reverse transitions release energy. The transitions include: melting/freezing between solid and liquid, vaporization/condensation between liquid and gas, ionization/deionization between gas and plasma, and sublimation/deposition as a direct path between solid and gas. In fluid mechanics, the liquid and gas phases are of primary interest, as both are considered fluids capable of flowing and deforming under applied forces.

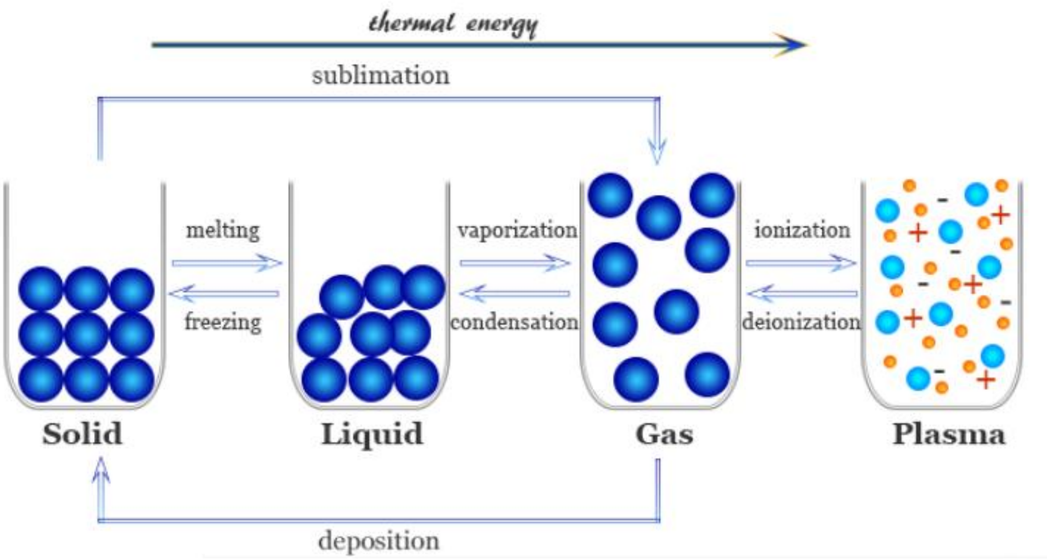


Figure 2-2 Phase changes between states. Energy is needed for changes from left to right, and is given back when the change is from right to left.

2.2 Definition of a fluid

Fluids are substances that are capable of flowing and conform to the shape of containing vessels. When in equilibrium, fluids cannot sustain tangential or shear forces. All fluids have some degree of compressibility and offer little resistance to change of form. Fluids can be classified as liquids or gases. The chief differences between liquids and gases are (a) liquids are practically incompressible whereas gases are compressible and usually must be so treated and (b) liquids occupy definite volumes and have free surfaces whereas a given mass of gas expands until it completely fills any containing vessel.

2.3 Mixed states

Not all fluids are pure liquid or gas. In some fluids, two phases in thermodynamic equilibrium coexist.

In contrast to pure liquids, the presence of particles (gas bubbles, solid particles, droplets) induces a multitude of interfaces between the liquid (continuous phase) and the particles (dispersed phase), which can radically change the nature of the mixture.

Dispersions are heterogeneous mixtures of at least two immiscible phases. They are divided into a dispersed phase and a continuous phase. The dispersed phase is made up of smaller particles or droplets distributed throughout the continuous phase. The continuous phase is generally present in larger quantities and surrounds the dispersed components (see Figure 2-3). Many industrial products are dispersions, such as milk, paint emulsions and creams.

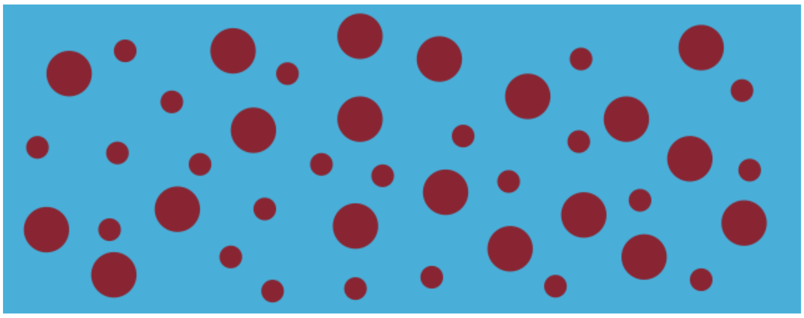


Figure 2-3 Illustration of a dispersion. The continuous phase is in blue and the dispersed one in red.. (dataphysics-instruments.com/, consulted on 2024-09-30)

Depending on the states of the continuous and dispersed phases, various types of dispersions can be defined (Table 2-2).

Table 2-2: Dispersion types according to the state of matter of the dispersed and continuous phases.

Type of dispersion	**Continuous phase**	**Dispersed phase**
Emulsion	Liquid	Liquid
Foam	Liquid	Gazeous
Suspension	Liquid	Solid
Fog	Gazeous	Liquid
Smoke	Gazeous	Solid
Wet porous solid	Solid	Liquid
Dry porous solid	Solid	Gazeous
Solid mixture	Solid	Solid

Dispersions can also be classified according to particle size. Mixtures with smaller particles are generally more stable.

- Molecularly dispersed systems contain particles smaller than 1 nm.
- Colloidal dispersions contain particles between 1 nm and 1 µm.

- Coarse dispersions contain particles larger than 1 µm.

2.4 Fluid characteristics

Any fluid can be characterized by several characteristics, in particular its density (which can be expressed in several ways), its elasticity and its thermal expansion and its viscosity.

2.4.1 Mass density ρ

Mass density, ρ (rho), gives the ratio of mass to volume at a point. Select a point (x, y, z) in space and select a small volume ΔV surrounding this point. The mass of the matter within the volume is Δm, and the density is:

$$\rho = \frac{Mass}{Volume} = \frac{\Delta M}{\Delta V} \quad \left[\frac{Kg}{m^3}\right]$$

The reciprocal of density is the specific volume v, which is defined as volume per unit mass, that is:

$$\vartheta = \frac{\Delta V}{\Delta M} = \frac{1}{\rho} \quad \left[\frac{m^3}{Kg}\right]$$

Fluid density is temperature dependent and to a lesser extent it is pressure dependent. For example the density of water at sea-level and 4°C is 1000 kg/m^3 , whilst at 50°C it is 988 kg/m^3 The density of most gases is proportional to pressure and inversely proportional to temperature. Liquids and solids, on the other hand, are essentially incompressible substances, and the variation of their density with pressure is usually negligible.

2.4.2 Specific gravity or relative density

Specific gravity is defined as the ratio of the density of a substance to the density of some standard substance at a specified temperature (usually water at 4°C, for which ρH_2O = 1000 kg/m^3). That is,

$$d = \frac{\rho_{liquid}}{\rho_{H2O}}$$

Note that the specific gravity of a substance is a dimensionless quantity.

2.4.3 Specific weight or weight density

Specific weight is represented by the Greek symbol γ (gamma). The weight of a unit volume of a substance is called specific weight or weight density and is expressed as:

$$\gamma = \frac{weight}{volume} = \frac{P}{V} \quad \left[\frac{N}{m^3}\right]$$

To relate γ and ρ, recall that weight and mass are related by $P = Mg$, where g is the acceleration from the terrestrial attraction. Divide this equation by volume to give:

$$\gamma = \frac{P}{V} = \frac{Mg}{V} = \rho g$$

At constant temperature, the densities of liquids do not vary much with the applied pressure, and thus they can often be approximated as being incompressible substances during most processes without sacrificing much in accuracy.

Based on observations, we know that when the temperature or pressure of a fluid is altered, its volume (or density) also changes. Generally, fluids expand when heated or when pressure is reduced, and they contract when cooled or when pressure is increased. However, different fluids experience different amounts of volume change under these conditions. To quantify these relationships between volume changes and variations in pressure and temperature, we use two specific properties: the bulk modulus of elasticity κ and the coefficient of volume expansion β.

2.4.4 The Bulk Modulus of Elasticity, compressibility

The bulk modulus of elasticity represents the change in pressure corresponding to a fractional change in volume or density of the fluid while the temperature remains constant.

$$E = \frac{\Delta P . V}{\Delta V}$$

The inverse of the coefficient of compressibility is called the isothermal compressibility ε and is expressed as

$$\varepsilon = \frac{\Delta V}{V \, \Delta p}$$

The isothermal compressibility of a fluid is the fractional change in volume or density corresponding to a unit change in pressure

Nota Bene : Constant density means that the density of a flowing fluid can be assumed to be constant spatially and temporally without causing significant changes (say 5%) in numbers that are calculated. Because liquids have a high value of bulk modulus, they are commonly assumed to be incompressible. Incompressible means that the density of each fluid particle is independent of pressure. A fluid that is incompressible can still have a variable density, meaning that density differs at various points in space or time.

2.4.5 Thermal expansion coefficient

The thermal expansion coefficient (also called the coefficient of thermal expansion) is a measure of how the volume of a fluid changes with respect to temperature change at constant pressure. It represents the relative volume change of a fluid per degree of temperature change. It is typically denoted by β (beta) and expressed in units of K^{-1} or $°C^{-1}$.

The mathematical expression is:

$$\beta = \frac{1}{V}\frac{\Delta V}{\Delta T} \quad [K^{-1}]$$

Where:

- V is the volume of the fluid $[m^3]$
- T is the temperature [K]
- p indicates that pressure remains constant [Pa]
- $\frac{\Delta V}{\Delta T}$ represents the partial derivative of volume with respect to temperature
- β (beta) is the volumetric thermal expansion coefficient $[\frac{1}{K}]$

2.5 Fluid incompressible

A fluid is said to be incompressible when the volume occupied by a given mass does not vary as a function of external pressure (constant density). Liquids can be considered incompressible fluids (water, oil, ... etc.).

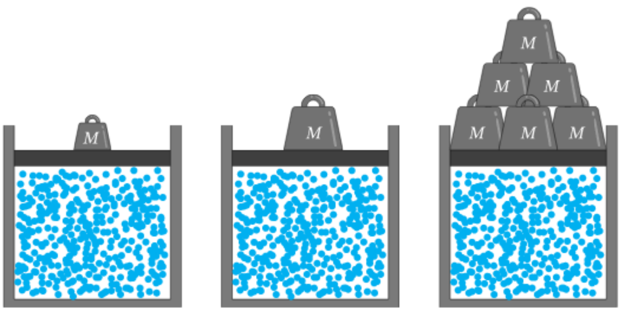


Figure 2-4 Incompressible fluid — volume remains nearly constant under increasing pressure (liquid)

2.6 Fluid compressible

Compressible fluid A fluid is said to be compressible when the volume occupied by a given mass varies as a function of external pressure (variable density). Gases are compressible fluids. For example, air, hydrogen and methane in their gaseous state are considered compressible fluids.

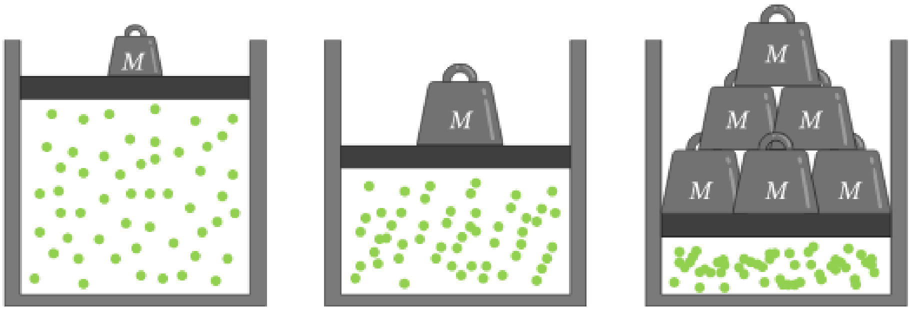


Figure 2-5 Compressible fluid — volume decreases significantly under increasing pressure (gas)

2.7 Ideal Fluid :

In fluid mechanics, a fluid is considered "perfect" when its motion can be described without considering the effects of friction. This concept of a perfect fluid is a simplified model used to facilitate calculations and is not practically found in nature. It serves as an idealized representation to understand fluid dynamics.

2.8 Real fluids,

on the other hand, account for the tangential forces of internal friction that oppose the relative sliding of different fluid layers in motion. This phenomenon is known as viscous friction. When studying the motion of real fluids, it is essential to consider these frictional forces. However, when a real fluid is at rest, it can be assumed to behave like a perfect fluid, as the effects of friction are negligible in this state.

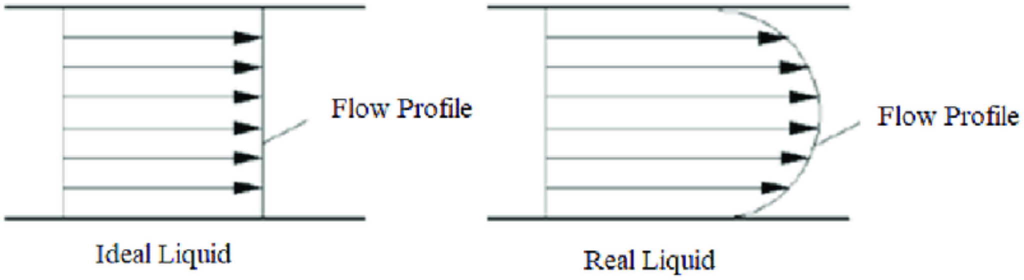


Figure 2-6: Velocity distribution in a tube with an ideal fluid and a real fluid.

2.9 Rheology:

Rheology is the branch of fluid mechanics that studies the deformation and flow behavior of materials, particularly fluids, under the influence of applied forces. It specifically examines the relationship between stress (force per unit area) and strain (deformation) in materials that exhibit both solid-like and fluid-like characteristics. The rheological properties of a fluid determine how it responds to applied forces and shear rates, which can include behaviors such as:

- viscosity (resistance to flow)
- elasticity (ability to return to original shape)
- plasticity (permanent deformation)
- thixotropy (time-dependent changes in viscosity)

2.9.1 Viscosity:

The viscosity of a fluid is that property which determines the amount of its resistance to a shearing force. Viscosity is due primarily to interaction between fluid molecules. Viscosity is a measure of the “stickiness” or “resistance to deformation” of a fluid. It is due to the internal frictional force that develops between different layers of fluids as they are forced to move relative to each other. Viscosity is caused by the cohesive forces between the molecules in liquids, and by the molecular collisions in gases. In general, liquids have higher dynamic viscosities than gases.

It appears that there is a property that represents the internal resistance of a fluid to motion or the “fluidity,” and that property is the viscosity. The force a flowing fluid exerts on a body in the flow direction is called the drag force, and the magnitude of this force depends, in part, on viscosity.

To obtain a relation for viscosity, consider a fluid layer between two very large parallel plates (or equivalently, two parallel plates immersed in a large body of a fluid) separated by a distance ℓ (Figure 2-7). Now a constant parallel force F is applied to the upper plate while the lower plate is held fixed. After the initial transients, it is observed that the upper plate moves continuously under the influence of this force at a constant speed V. The fluid in contact with the upper plate sticks to the plate surface and moves with it at the same speed, and the shear stress τ acting on this fluid layer is:

$$\tau = \frac{F}{A}$$

where A is the contact area between the plate and the fluid. Note that the fluid layer deforms continuously under the influence of shear stress. The fluid in contact with the lower plate assumes the velocity of that plate, which is zero (because of the no-slip condition).

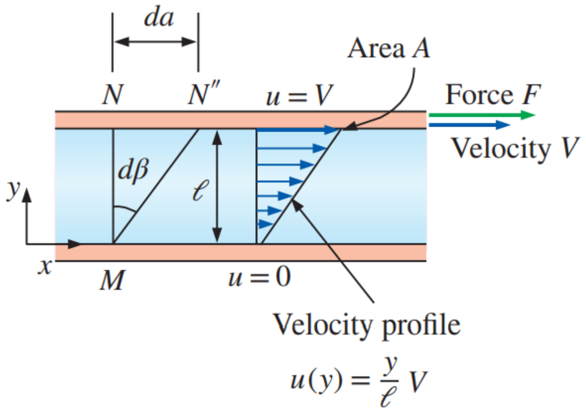


Figure 2-7: The behavior of a fluid in laminar flow between two parallel plates when the upper plate moves with a constant velocity.

The deformation of a fluid element is equivalent to the velocity gradient *du/dy*. Further, it can be verified experimentally that for most fluids the rate of deformation (and thus the velocity gradient) is directly proportional to the shear stress.

Fluids for which the rate of deformation is linearly proportional to the shear stress are called Newtonian fluids after Sir Isaac Newton, who expressed it first in 1687. Most common fluids such as water, air, gasoline, and oils are Newtonian fluids. Blood and liquid plastics are examples of non-Newtonian fluids. In one-dimensional shear flow of Newtonian fluids, shear stress can be expressed by the linear relationship.

$$\tau = \mu \frac{du}{dy}$$

Where:

μ : constant of proportionality called the coefficient of viscosity or the dynamic (or absolute) viscosity of the fluid, whose unit is kg/m·s, or equivalently, N·s/m^2 (or Pa·s where Pa is the pressure unit pascal).

The Poise (P) is a unit of dynamic viscosity, named after Jean Louis Marie Poiseuille, a French physicist and physiologist.

1 Poise = 1 g/(cm·s) = 0.1 Pa·s

2.9.2 Newtonian and non-Newtonians fluids

A Newtonian fluid is a fluid in which the shear stress varies linearly with the velocity gradient (shear rate), meaning its dynamic viscosity remains constant regardless of the applied stress — water and air are typical examples. A non-Newtonian fluid, on the other hand, exhibits a viscosity that changes with the applied shear rate; depending on their behavior, these fluids are classified as shear-thinning (pseudoplastic, n<1), shear-thickening (dilatant, n>1), or Bingham plastic, as illustrated in Figure 2-8.

Figure 2-9 illustrates how increasing viscosity significantly reduces the flow velocity of a fluid, as demonstrated by honey pouring from a spoon — the higher the viscosity, the slower and more resistant the flow.

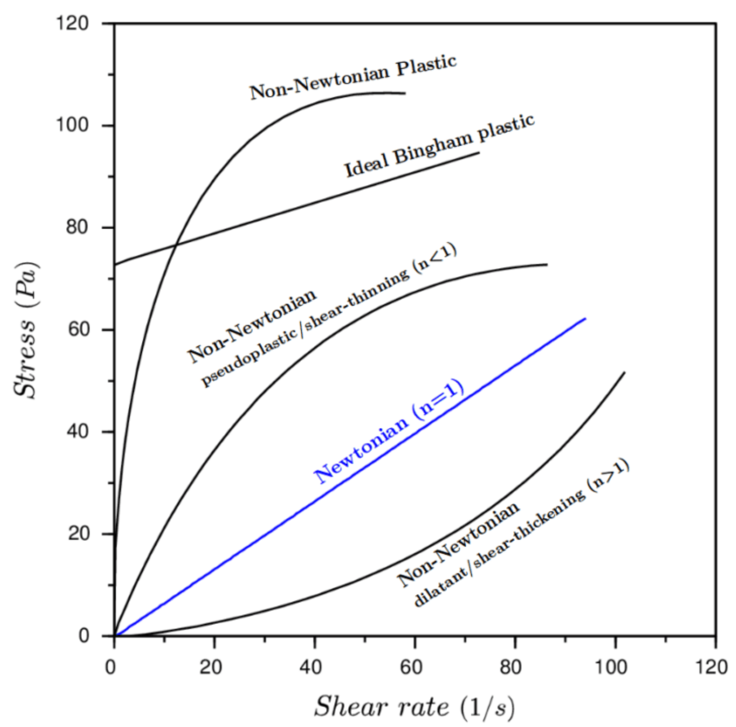


Figure 2-8 : Newtonian and Non-Newtonian Fluids

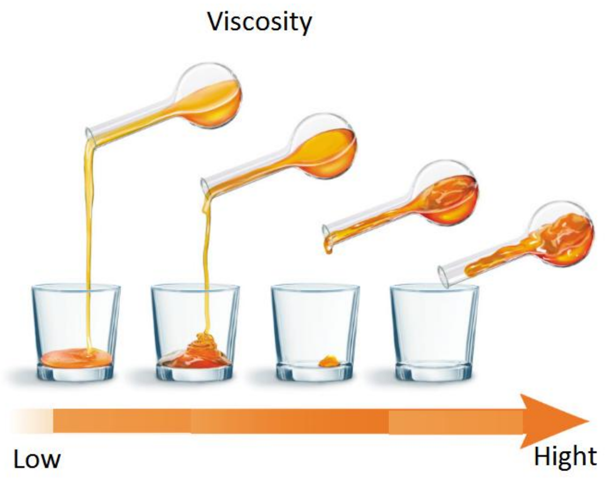


Figure 2-9 : Effect of an increasing viscosity on the flow velocity.

2.9.3 Dynamic and kinematic viscosity

In fluid mechanics and heat transfer, the ratio of dynamic viscosity to density appears frequently. For convenience, this ratio is given the name kinematic viscosity ν. It represents the internal resistance to flow under gravity, characterizing a fluid's ability to dissipate momentum through internal molecular motion. It is analogous to the diffusion of momentum within the fluid.

$$\nu = \frac{\mu}{\rho} \qquad \left[\frac{m^2}{s}\right]$$

The key difference between dynamic and kinematic viscosity is:

- Dynamic Viscosity (μ): Measures the fluid's resistance to flow under an applied force or shear stress. It relates to the force required to cause fluid motion.
  - Used when dealing with force calculations
  - Important in pump and pipeline design
- Kinematic Viscosity (ν): Measures the fluid's resistance to flow under the influence of gravity. It relates to fluid's natural flow behaviour under gravity.
  - Used in free surface flow calculations
  - Important in Reynolds number calculations
  - Natural convection studies

2.9.4 Properties of Viscosity in Fluids

For Liquids:

- Viscosity decreases with increasing temperature
- Viscosity increases with increasing pressure (though this effect is relatively small)
- Viscosity is typically much higher than in gases

For Gases:

- Viscosity increases with increasing temperature (unlike liquids)
- Viscosity is relatively independent of pressure at moderate pressures
- Viscosity is generally much lower than liquids

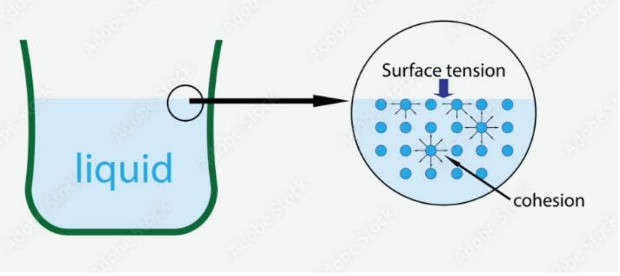
2.9.5 Surface tension

Liquid molecules are mutually attracted by forces called cohesive forces. The forces of attraction between the molecules of two different immiscible liquids (or liquid materials) are called adhesive forces.

A liquid molecule at rest is subject to the surface contact forces exerted on it by neighboring molecules, with an average value of zero.

A molecule at the free surface of a liquid or at the interface between two liquids is no longer subject to the action of symmetrical forces, since it is no longer symmetrically surrounded by other molecules of the same type.

As a result, the molecular forces are no longer zero, and cause "surface tension" in a direction normal to the separating surface.



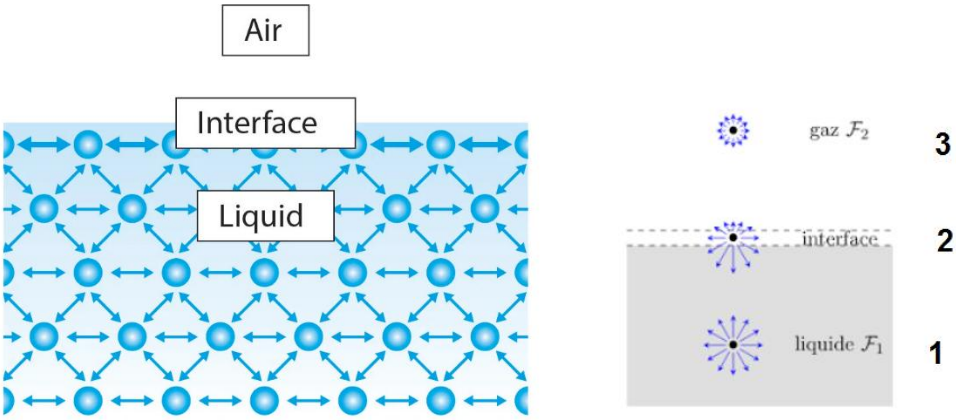


Figure 2-10: Intermolecular forces in: 1 a liquid, 2 at the interface and 3 in a gas.

A molecule at the free surface, or in the zone of separation between two fluids, has an energy corresponding to the work done by the molecule to reach the surface. The free surface behaves like a taut membrane.

σ is the tension per unit length of any line on the separation surface.

$$\sigma = [M.L^2] = \frac{Kg}{m^2} = N/m$$

Table 2-3: Surface tension for some liquids in contact with air.

Liquid	$\sigma(N.m^{-1})$ at 20 °C
Water at 20°C	$73\cdot 10^{-3}$
Water at 0°C	$75.6\cdot 10^{-3}$
vegetable oil	$32\cdot10^{-3}$
Ethanol	$22\cdot10^{-3}$
Ether	$17\cdot 10^{-3}$
Mercury	$480\cdot10^{-3}$

Therefore, we can say that surface tension is a property of fluids, which are attracted or repelled when in contact with another solid, liquid or gas.

This property explains various phenomena shown in Figure 2-11.

The formation of raindrops.	The meniscus around the edges of a tube.	The levitation of insects on the surface of water (some insects are able to move on water).

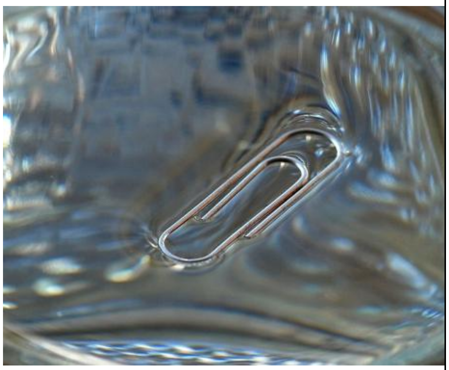
		
The floating paper clip 	A floating cent 	“Sticking” water 

Figure 2-11: Examples of the effects of surface tension.

2.9.6 Contact Angle (Wettability):

Molecules on a solid surface attract liquid molecules with a that differs from the force between liquid molecules. When greater, the liquid tends to wet the surface, like water on clean glass or metal. When smaller the liquid tends to retract or form droplets on the surface, like mercury on glass or water on greasy surfaces (Figure 2-12).

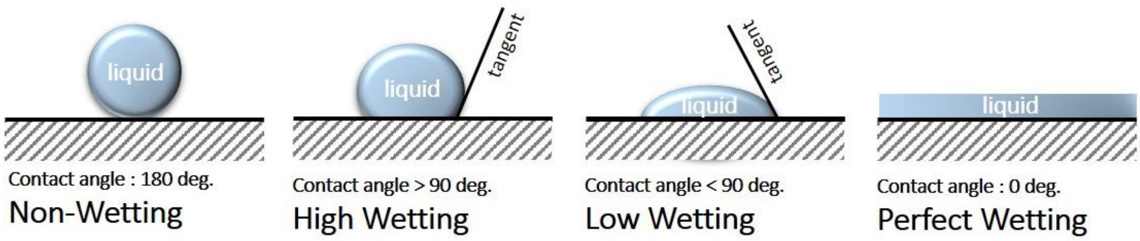


Figure 2-12: Effects of liquid-solid interface tension.

When a drop of liquid is deposited on a flat solid surface, the contact angle (θ) is defined as the angle between the tangent to the liquid-vapor interface at the triple point (where solid, liquid, and vapor meet) and the solid surface.

The contact angle θ depends on the interactions between three phases: the liquid, the solid, and the surrounding gas/vapor. Three interfacial tensions must be considered:

- γ_{sl}: Surface tension between solid and liquid
- γ_{lv}: Surface tension between liquid and vapor
- γ_{sv}: Surface tension between solid and vapor

The contact angle for a given three-phase system does not change with the macroscopic shape of the solid.

When a drop of liquid is deposited on a flat, horizontal solid surface, three scenarios can occur:

1. Complete wetting ($\theta = 0°$): The liquid spreads completely over the solid surface
2. Partial wetting ($0° < \theta < 90°$): The liquid partially wets the solid surface
3. Non-wetting ($\theta > 90°$): The liquid tends to minimize contact with the solid surface, forming a more spherical droplet

2.9.7 Capillarity

In a narrow glass tube, the interaction between the liquid and the surface creates a curved interface called a meniscus. For wetting liquids like water in glass, the meniscus is concave and the liquid rises along the walls due to capillary action.

Capillary action at the free surface of a liquid in a narrow tube is due to surface tension. If the liquid wets the wall, the free surface is raised, forming a concave meniscus (Figure 2-13 a); if the liquid does not wet the wall, the free surface is lowered, forming a convex meniscus (Figure 2-13 b). The phenomenon of raising and lowering the free surface in the tube is known as capillarity.

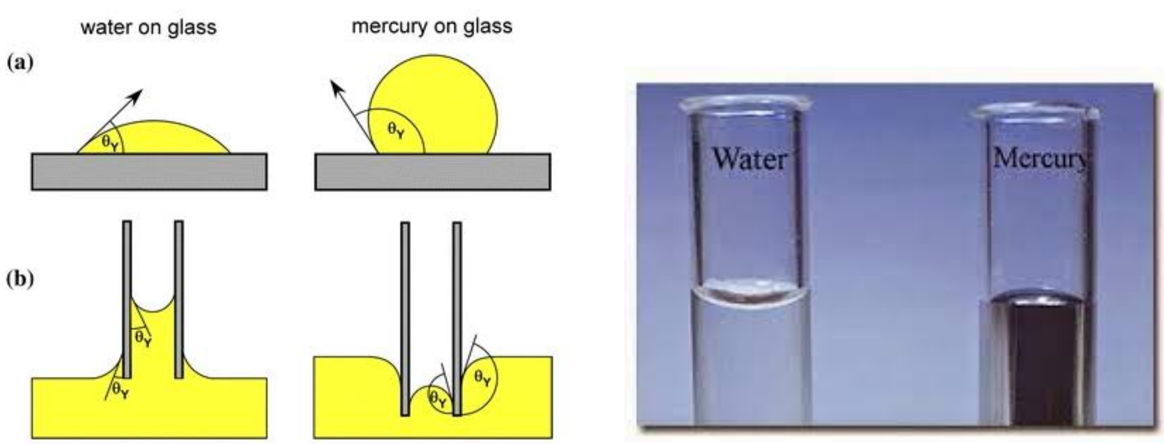


Figure 2-13: Examples of a) sessile drop and b) capillary rise of water and depression of mercury on the same glass surface. Water rises, while mercury sinks in a narrow tube.

The raising of water in the tube is called "capillary rise", and the lowering of mercury is called "capillary depression".

The capillarity value depends on the tube diameter, the density of the liquid and its surface tension. The height of capillarity "h" can be obtained from the following relationship (Jurin's law, Figure 2-14) :

$$h = \frac{(4cos\theta)\sigma}{\gamma d}$$

θ : is the contact angle between the liquid surface and the solid plane,

d : is the tube diameter.

γ : weight density of the liquid [N/m^3]

σ: surface tension of the liquid [N/m]

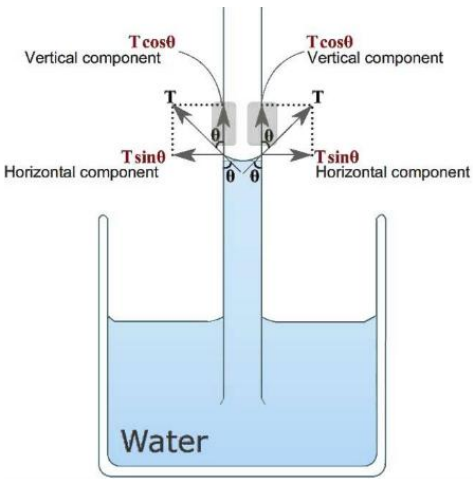


Figure 2-14: Capillarity ascension in a tube

PRACTICAL EXERCISES

Exercise N°1:

Oil is poured into a cylindrical tank that is 4 meters tall and has an internal diameter of 2 meters. The oil reaches a height of 3.6 meters. When weighed, the oil has a mass of 9772 kilograms. Determine the density of this oil.

Solution N°1 :

By definition, density is given by the following relationship

$$\rho = \frac{M}{V}$$

Where V is the volume of the oil.

$$V = S.H$$

$$V = \frac{\pi D^2}{4} H = \frac{\pi (2)^2}{4} . 3.6 = 11.304 m^3$$

Then:

$$\rho = \frac{9772}{11.304} = 864.47\, Kg/m^3$$

Exercise N° 2:

Given that 2 liters of paraffin oil weigh 1.6 kilograms and the acceleration due to gravity is 9.81 m/s², calculate the following:

1. Volumetric mass density of the oil
2. Specific weight of the paraffin oil
3. Specific gravity of the oil

Solution N° 2:

Oil density:

$$\rho = \frac{M}{V} = \frac{1.6}{2.10^{-3}} = 800\, Kg/m^3$$

Specific weight of oil

$$\gamma = \rho . g = 800.9.81 = 7848 \ N/m^3$$

Specific gravity of the oil

$$d = \frac{\rho_{Oil}}{\rho_{H2O}} = \frac{800}{10^3} = 0.8$$

Exercise N° 3:

An industrial tank contains exactly 5 m^3 of oil. The total weight (tank + oil) is 5122 kg. If the empty tank weighs 962 kg, determine:

a) The density of the oil

b) The specific gravity of the oil, given that the density of water is 1000 kg/m^3.

Solution N°3:

Oil density:

$$\rho = \frac{M}{V} = \frac{(5122 - 962)}{5} = 830 \text{ kg/m}^3.$$

Specific gravity of the oil

$$d = \frac{\rho_{Oil}}{\rho_{H2O}} = \frac{830}{10^3} = 0.83$$

Exercise N°4:

1- Find the coefficient of expansion of a fluid with an initial volume of 20 liters, knowing that a temperature increase of 20°C results in a volume increase of 0.1 liter.

2- Find the change in volume of 30 liters of water at 20°C for a pressure increase of 20bars. (E at 20°C is given as 2.25 10^9 Pa)

3- Determine the bulk modulus of elasticity of water, knowing that at 40 bar the volume is 30 liters and at 246.8 bar the volume is 29.73 liters?

Solution N° 4:

1. The coefficient of volume expansion is given by the following formula:

$$\beta = \frac{\Delta V}{V}\frac{1}{\Delta T} = \frac{0.1.10^{-3}}{20.10^{-3}}\frac{1}{20} = 0.25.10^{-3}K^{-1}$$

Caution!!!!! : $\Delta T = 20°C = 20K$

We can find the change in volume using the compressibility formula, which is related to the bulk modulus of elasticity as:

$$E = \frac{\Delta P.V}{\Delta V} = \frac{1}{\varepsilon} \text{ (selon le chapitre, § 2.4.4)}$$

$$\Delta V = \frac{V}{E}\Delta P = \frac{30.10^{-3}}{2.25\ 10^{9}}.20.10^{5} = 0.266\ .10^{-4} m^{3}$$

2. bulk modulus of elasticity of water is given as follows:

$$E = -\frac{\Delta P}{\Delta V}V = -\frac{(246.8-40)10^{5}}{(29.73-30)10^{-3}}30.10^{-3} = 2.29.10^{9}\ Pa$$

Exercise N°5:

Water at 30°C is able to climb up a clean glass of 0.2-mm-diameter tube due to surface tension. The water-glass angle is 0 – with the vertical ($\theta \approx 0$ in Fig. 3.7). How far up the tube does the water climb?

Solution N°5:

The height that the water climbs is given by Eq. (1.22). It provides

$$h = \frac{4\sigma\ cos\theta}{\gamma D} = \frac{4\times0.0718\times1}{1000\times9.81\times0.0002} = 0.147 \text{ m or } 14.7 \text{ cm}$$

Exercise N° 6:

Determine the approximate capillary depression of mercury within a 1.5 mm diameter capillary tube at 20°C. Given that the surface tension of mercury is 0.515 N/m at 20°C and its density is 13570 kg/m³

Solution N°6:

by applying Jurrin's law

$$h = \frac{4\sigma cos\theta}{\gamma d}$$

therefore:

$$h = \frac{4 \cdot 0.515 \cdot \cos{(130)}}{9.81.13570.1.5.10^{-3}} = 6.63.10^{-3} m \text{ or } 6.63 \text{ mm}$$

Exercise N°7:

If the weight density of a liquid is 8.1 kN/m^3 , what is its relative density? The mass density of water is 1000 kg/m^3

Solution N°7:

The mass density of the liquid is 8100 /9.81 =825.7 kg/m^3, where 9.81 m/s^2 is the acceleration from the earth attraction.

The relative density is hence 825.7/1000 =0.8257

Exercise N°8:

Calculate the density of 1 liter of liquid weighing 7 N.

Solution N°8:

The weigth density is $7/10^{-3}$ = 7000 N/m^3,

The mass density is 7000/9.81 = 713.6 kg/m^3

The relative density is 0.7136 if we take 1000 kg/m^3 for the mass density of water.

Exercise N°9:

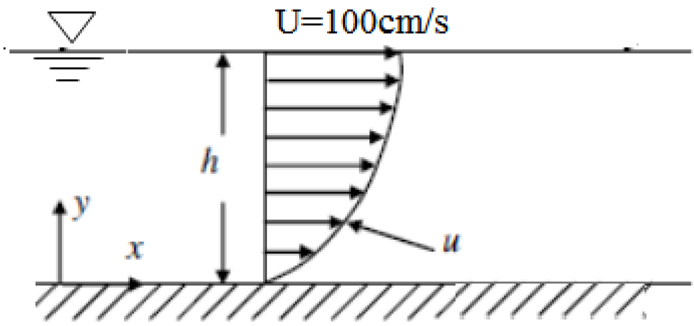
Calculate the mass of 500 cm^3 of liquid if the density is 12.4 k N/m^3 .

Solution N°9:

The mass density of that liquid is 12400 /9.81 = 1264 kg/m^3, hence the mass of 500 cm^3 is $1264 \cdot 500 \cdot 10^{-6}$ = 0.632 kg

Exercise N°10:

A fluid with an absolute viscosity of 8.2 10^{-2} $N.s/m^2$ flows through a 15 cm depth large open channel as shown in the figure. Calculate the velocity gradient, the flow velocity and the tangential stress intensity at the wall and at points 5, 10 and 15 cm from it, assuming a parabolic velocity distribution and laminar flow. What can we conclude?



Solution N°10:

The velocity has a parabolic distribution, so its equation has the form :

$$v(y) = Ay^2 + By + C$$

Where the constants A, B and C are to be determined from boundary conditions given in the problem as:

$y = o \implies v(0) = 0$ (no slip at the plate surface), hence C = 0

At $y = 15\ cm \implies v(15cm) = 100cm/s$. Hence 100 = A 15^3+B· 15

And the velocity gradient: $\frac{dv}{dy} = 0$ for $y = 15\ cm$ (condition for the vortex of parabola, maximum velocity of the function).

Since : $\frac{dv}{dy}(y) = 2Ay + B,\ 0 = 2\ A \cdot 15 + B$

From these two equations, we get B = -30 A, hence A = -100/225 = -0.444 and B = 13.333 if y is expressed in cm and velocity in cm/s.

The equation becomes: $v(y) = -0.444y^2 + 13.33y$

$\frac{dv}{dy}(y) = -0.888y + 13.33$

And $\tau(y) = 8.2\ 10^{-4}\ \frac{dv}{dy}(y)$ (the velocity is in cm/s and the viscosity is in Ns/m²

The results for abscissas y=0, 5, 10 and 15 are given in the following table:

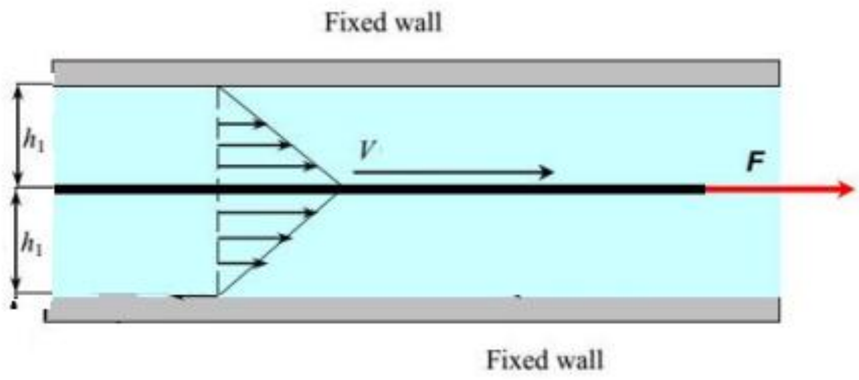
Y	$v(y)(cm/s)$	$\frac{dv}{dy}(y)(s^{-1})$	$\tau(y)(Pa)$
0	0	13.33	1,09E-02
5	55.5	8.89	7,29E-03

10	88.9	4.44	3,65E-03
15	100	0	0,00E+00

We can conclude that where velocity is zero, tangential stress is maximum; and where velocity is maximum, tangential stress is zero (which occurs at the centerline of a pressurized pipe).

Exercise N°11:

A thin solid plate of area A=0.75 m² and negligible weight is traced horizontally inside a film of oil of thickness h=2.5 cm and dynamic viscosity μ=0.785 Pa s. The velocity distribution is assumed to be linear.



What is the value of the force F required to apply to the plate to impart a linear velocity of V=0.5m/s?

a- If the plate is located on the centerline of the film

b- If it is located 1cm from one of the two solid walls

Solution N°11 :

The force required to be applied to the plate = the sum of the forces exerted on both sides of the plate.

first case:

If the plate is located on the center line of the oil film the friction force is the same on both sides of the plate by symmetry, and the total traction force is hence twice the force on each side

$$F1 = 2\,\tau_1 x A$$

$$F1 = 2\,\mu\left(\frac{dv}{dy}\right)_1 \times A = 2 \times 0.785\left(\frac{0.5}{1.25 \times 10^{-2}}\right) \times 0.75 = 47\ \ N$$

Case 2:

If the plate is located 1cm from one of the two solid plates

The friction force on the upper side of the plate :

$$F1 = \tau_1 \times\ A$$

$$F1 = \mu\left(\frac{dv}{dy}\right)_1 \times A = 0.785\left(\frac{0.5}{1.5 \times 10^{-2}}\right) \times 0.75 = 19.6\ \ N$$

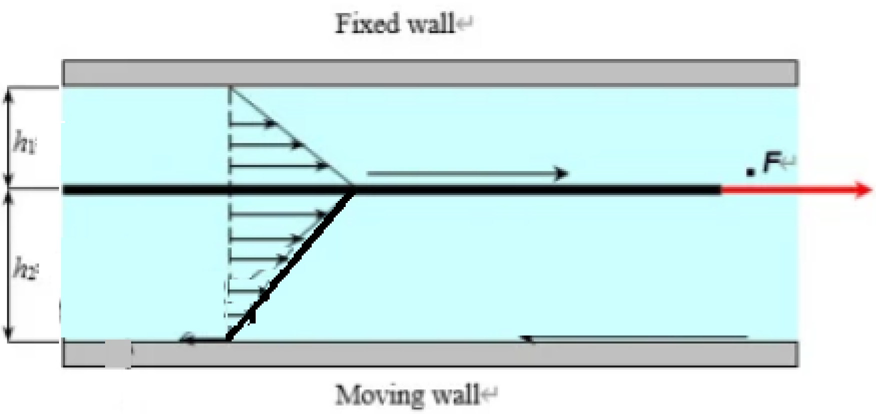
the friction force on the underside of the plate :

$$F2 = \tau_2 x\ A$$

$$F2 = \mu\left(\frac{dv}{dy}\right)_2 \times A = 0.785\left(\frac{0.5}{1.\times 10^{-2}}\right) \times 0.75 = 29.4\ \ N$$

And so the force F :

$$F\ = F1 + F2 = 49\ N$$



Chapter 2: Fluid statics

Chapter Objectives

By the end of this chapter, students should be able to:

- ✓ Derive The Fundamental Equation Governing Fluid Statics;
- ✓ Determine The Pressure Distribution Within A Fluid At Rest;
- ✓ Evaluate The Hydrostatic Forces Exerted By A Stationary Fluid On Rigid Surfaces;
- ✓ Locate The Point Of Application Of The Resultant Hydrostatic Force.

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1 Introduction:

Hydrostatics is the science of liquid equilibrium. In particular, it studies the transmission of pressure. Fluid statics is the science that studies the equilibrium conditions of fluids at rest. More precisely, it concerns all situations in which there is no relative motion between fluid particles: ! fluids at rest ! uniformly accelerated fluids There are no constraints due to friction between particles. The only forces at play are volume forces due to weight and surface forces due to pressure. In hydrostatics, with the fluid at rest, the laws established for a perfect fluid will apply to a real fluid. A real fluid differs from an ideal fluid in its viscosity, which only manifests its effects when there is displacement.

2 Forces on fluid element

An infinitesimal region of fluid continumm can be defined as a fluid elment.

A fluid element, in isolation from its surroundings, is experienced by tow types of extrenals forces:

• **Body forces:** these forces act throughout the body of the fluid elment and are distributed over the entire mass or volume of the element. Long range forces like gravity, electromagnetic forces, fictitious forces such as centrifugal forces, etc. Body forces ate usually expressed per unit mass of element or medium upon which the forces act.

• **Surface forces:** they include all forces exeted on fluid element by its surrondings through direct contact at the surface . therefore, these forces appear pnly at the surface of fluid element.

Short range forces such as viscosity, surface tension, etc.

3 Pressure at a Point

A fundamental question in fluid mechanics is how pressure varies with the orientation of a surface at a given point within the fluid. To investigate this, consider a small triangular wedge of fluid isolated from its surroundings (Figure 2. 1). In the absence of shearing stresses (typical for ideal fluids), the only external forces acting on the wedge are due to pressure and gravity. For simplicity, we'll neglect forces in the x-direction and assume the z-axis points vertically upward. While we're primarily interested in fluids at rest (hydrostatics), we'll allow for accelerated motion to maintain generality. The key assumption of zero shearing stress remains valid as long as the fluid element moves as a rigid body, without internal deformation.

The equations of motion for the fluid element, derived from Newton's second law (F = ma), in the y and z directions are as follows:

$$\sum F_y = P_y \partial x \partial z - P_s \partial x \partial s \, sin\theta = \rho \frac{\partial x \partial y \partial z}{2} a_y$$

$$\sum F_z = P_z \partial x \partial y - P_s \partial x \partial s \, cos\theta - \gamma \frac{\partial x \partial y \partial z}{2} = \rho \frac{\partial x \partial y \partial z}{2} a_z$$

Where:

P_s, P_y, P_z: are the average pressures on the faces

γ , : are the fluid specific weight and density, respectively

a_y, a_z: the accelerations

To determine the force exerted by a pressure, it must be multiplied by the area over which it acts. Geometrically, we can see that :

$$\partial y = \partial s \, cos\theta$$

$$\partial z = \partial s \, sin\theta$$

Therefore, the equations of motion can be rewritten as:

$$P_y - P_s = \rho a_y \frac{\partial y}{2}$$

$$P_z - P_s = (\rho a_z + \gamma) \frac{\partial z}{2}$$

As we are primarily concerned with the infinitesimal behavior at a point, we take the limit as the dimensions ∂x, ∂y and ∂z tend to zero, while keeping the angle θ constant. This limiting process reveals that:

$$P_z = P_s$$

$$P_y = P_s$$

Or

$$P_z = P_y = P_s$$

The angle theta was chosen arbitrarily, so we can conclude that the pressure at a point in a fluid, whether at rest or in motion, is independent of direction as long as there are no shear stresses present.

This significant result is known as Pascal's law, named after Blaise Pascal (1623–1662), a French mathematician who made notable contributions to hydrostatics. As illustrated in the accompanying photograph, at the point where the side and bottom of the beaker meet, the pressure is the same on the side as it is on the bottom.

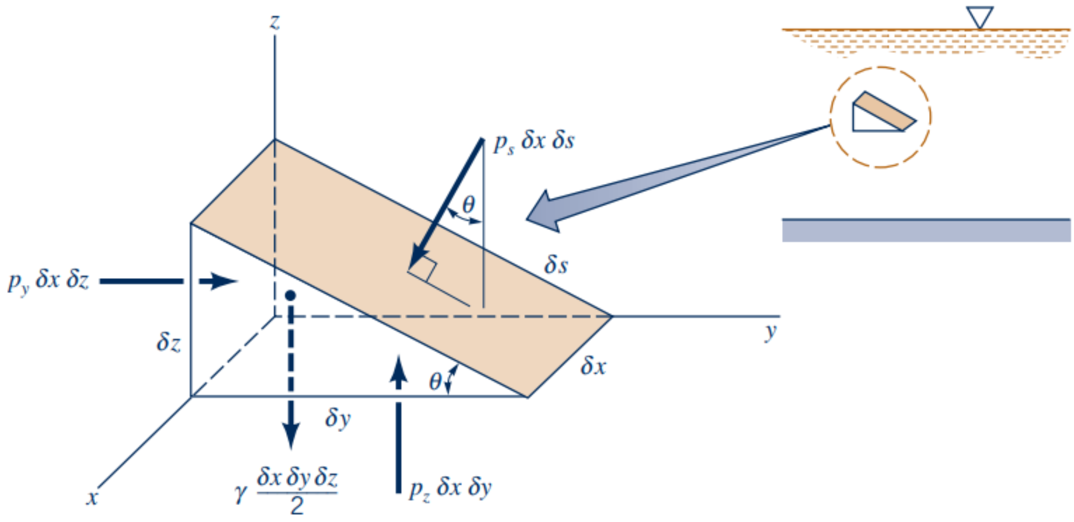


Figure 2. 1 Figure Forces on an arbitrary wedge-shaped element of fluid.

4 Fundamental Equation of Fluid Statics

Consider a small rectangular fluid element located within a larger fluid mass, as depicted in Figure. Two primary types of forces act upon this element: surface forces arising from pressure differentials, and body forces equivalent to the element's weight. We will not delve into other potential body forces, such as those induced by magnetic fields, in this text.

If we denote the pressure at the element's center as 'p', the average pressures on the various faces can be expressed in terms of 'p' and its derivatives, as illustrated in Figure. This involves a Taylor series expansion of the pressure at the element's center to approximate pressures at nearby locations, neglecting higher-order terms that diminish as the distances Δx, Δy, and Δz approach zero. A marginal figure clarifies this concept.

Let us consider, as shown in the diagram in Figure 2, a fluid volume element in the shape of a parallelepiped with a volume of dxdydz:

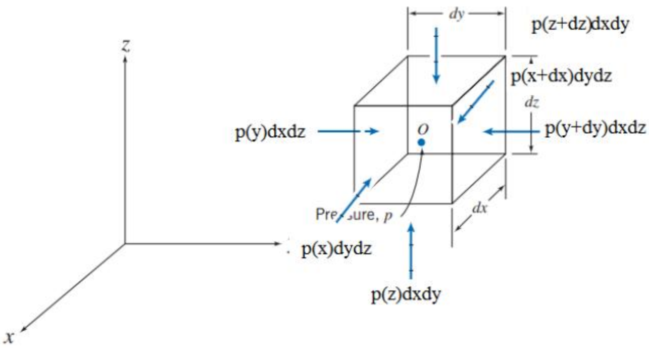


Figure 2. 2 Surface and body forces acting on small fluid element.

If we examine the forces acting on this small fluid volume, we can categorize them into two groups: surface forces and body forces. Surface forces, in this case, are solely due to pressure since we are limiting our analysis to fluids that are either stationary or moving with constant acceleration. Body forces, on the other hand, are represented by gravity, specifically the weight of the infinitesimal volume dV.

4.1 Surface forces

If we note the dFz component along z of the pressure force, it can be written as :

$dF_z = p(z)dxdx - p(z + dz)dxdy$

where p(z) and p(z+dz) are respectively the pressures acting on the bottom and top faces of the parallelepiped. Since dz is small, we can develop p(z+dz) to first order:

$p(z + dz) = p(z) + \left(\frac{\partial p}{\partial z}\right)_z dz$

As a result, it comes :

$$dF_z = -\left(\frac{\partial p}{\partial z}\right)_z dxdydz = -\left(\frac{\partial p}{\partial z}\right)_z dV$$

By analogy, following the other directions, we find :

$dF_x = -\left(\frac{\partial p}{\partial x}\right)_x dV$ et $dF_y = -\left(\frac{\partial p}{\partial y}\right)_y dV$

The resultant is therefore :

$$d\vec{F} = -\left(\frac{\partial p}{\partial x}\vec{e}_x + \frac{\partial p}{\partial y}\vec{e}_y + \frac{\partial p}{\partial z}\vec{e}_z\right)dV = -\vec{\nabla}p dV$$

4.2 Volume forces

This is simply the weight of the volume element dV. Thus :

$d\vec{P} = dm\vec{g} = \rho dV\vec{g}$

where g is the acceleration of gravity and rho the density of the fluid.

From the ***Fundamental Principle of Dynamics*** (FPD), we know that the resultant of the forces exerted on a body is equal to the product of its mass dm and the acceleration it undergoes $\vec{a}(x, y, z)$. Here, either the fluid is at rest, in which case$\vec{a}(x, y, z) = \vec{0}$, or it is uniformly accelerated, so we can write $\vec{a}(x, y, z) = \vec{a}\ \forall(x, y, z)$

We therefore have

$d\vec{F} + d\vec{P} = dm\vec{a}$ or even $-\vec{\nabla}p dV + \rho dV\vec{g} = \rho dV\vec{a}$

Simplifying by dV, we obtain a **local equation**, independent of the chosen volume element and therefore valid at any point of the fluid:

$-\vec{\nabla}p + \rho\vec{g} = \rho\vec{a}$

which, for a fluid at rest, simplifies to give the ***fundamental hydrostatic equation***:

$$\vec{\nabla}p = \rho\vec{g}$$

Let's consider that the gravity field is such that

$$\vec{g} = -g\vec{e_z}$$

By projection onto the three axes of the Cartesian reference frame, we obtain:

$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ and $\frac{\partial p}{\partial z} = -\rho g$

Consequently, $p(x, y, z) = p(z)$and the fundamental equation of hydrostatics can be summarized as :

$\frac{dp}{dz} = -\rho g$

$$-\nabla p + \rho g = 0$$

The physical meaning of each term is :

$$-\nabla p \qquad + \qquad \rho \vec{g} \qquad = 0$$

$$\begin{Bmatrix} Net\ pressure\ force \\ per\ unit\ volume \\ at\ a\ point \end{Bmatrix} + \begin{Bmatrix} body\ force \\ per\ unit\ volume \\ at\ a\ point \end{Bmatrix} = 0$$

This equation show that, under the given assumptions, pressure is solely a function of the z-coordinate, independent of x and y. As such, we can replace partial derivatives with total derivatives. With these simplifications, Equations reduce to...

$$\frac{dp}{dz} = -\rho g$$

It can be concluded that: Pressure increases linearly with depth

Before delving into practical applications, it's essential to understand that pressure measurements are always made relative to a specific reference level. If this reference level is a perfect vacuum, the resulting pressure is known as absolute pressure, as depicted in Figure
The majority of pressure gauges are designed to measure the pressure difference between the fluid being measured and the ambient pressure, typically atmospheric pressure. Pressures measured relative to atmospheric pressure are known as gauge pressures. Consequently :

Absolute pressure= gage pressure+ atmospheric pressure

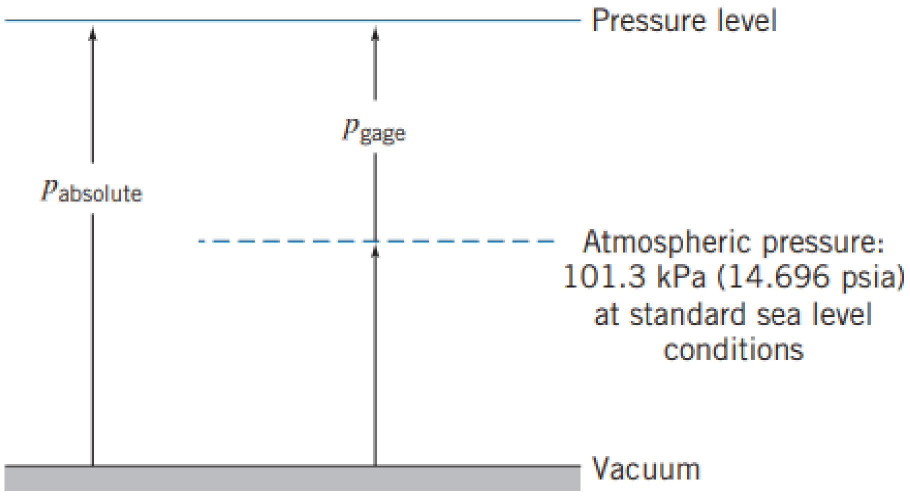


Figure 2. 3 Absolute and gage pressures, showing reference levels

5 Implications and Applications of the Fundamental Principle of Hydrostatics

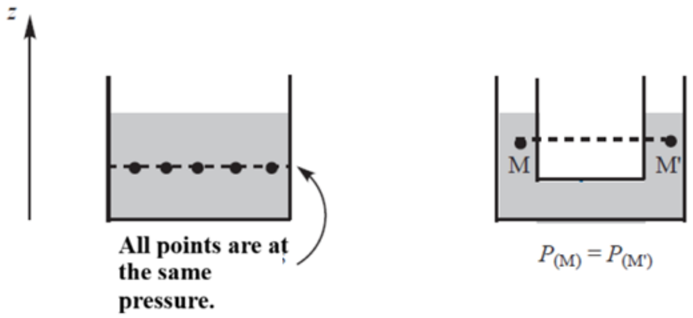
The fundamental law of hydrostatics states that the pressure difference between two points in a fluid at rest is proportional to their depth difference.

The main resulting consequences are as follows:

5.1 Level surface:

For a homogeneous fluid at rest, subjected only to the gravitational field, all points lying on the same horizontal surface have the same pressure.

In other words, any two points at the same elevation within the same fluid experience the same pressure; this is the case for points located along a horizontal line in the configurations shown on the figure below.



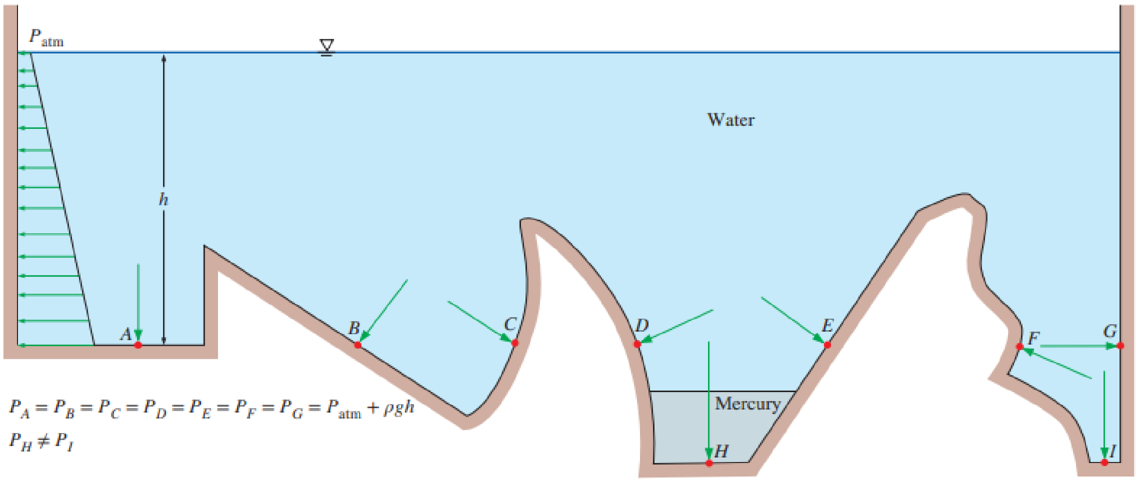


Figure 2. 4 : When a fluid is at rest (hydrostatic conditions), the pressure at any point on a horizontal plane within that fluid is identical, irrespective of the fluid's shape or configuration. This principle holds true as long as the points are interconnected by the same fluid.

The pressure at the bottom of containers of different shapes, with the same height and containing the same fluid, is identical.

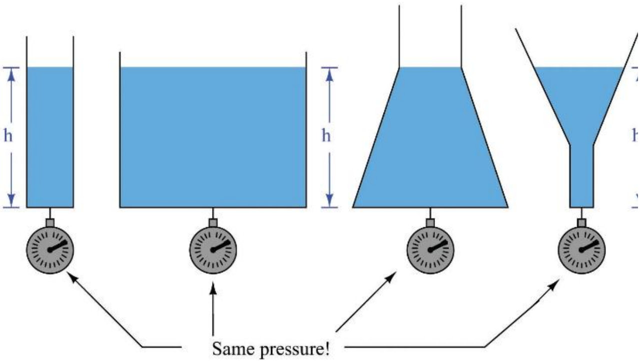


Figure 2. 5 Pressure vs. Container Shape.

5.2 Horizontal free surface:

For a homogeneous fluid at rest, subjected only to the gravitational field, the free surface is flat and horizontal. Indeed, all points on this surface are at the same pressure (equal to atmospheric pressure). This is why the surface of water in a container is always flat and horizontal.

5.3 Pressure for superimposed immiscible fluids:

We consider two immiscible fluids (I) and (II) (e.g. water and oil), with densities ρ_1 and ρ_2, in the same container (On the right of Figure 2. 6). Let A and B be two points on the separation surface, assumed (by contradiction) to be non-horizontal.

In the fluid (I) : $P_B - P_A = \rho_1 gh$

In the fluid (II) : $P_B - P_A = \rho_2 gh$

This implies that : $\rho_1 gh = \rho_2 gh \Longrightarrow (\rho_1 - \rho_2)gh = 0$

Given that $(\rho_1 - \rho_2) \neq 0$ and $g \neq 0$ then : $h = 0$

Conclusion : The separation surface of two immiscible liquids at rest is horizontal.

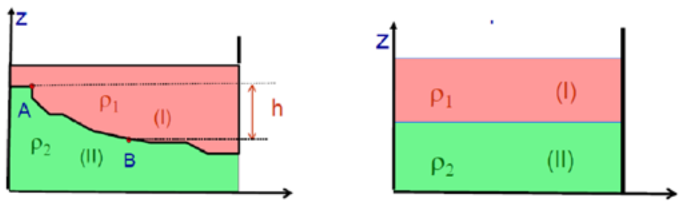


Figure 2. 6 surface of two immiscible liquids at rest

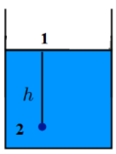
6 Pascal's law

Any change in pressure applied to an enclosed fluid is transmitted undiminished to all portions of the fluid and to the walls of its container. This principle, known as Pascal's Principle, was formulated by the French scientist Blaise Pascal in 1653. This phenomenon of pressure transmission is fundamental to the operation of hydraulic presses, jacks, and elevators.

<u>Note:</u> *An incompressible fluid at rest transmits pressure variations uniformly in all directions (rather than directly transmitting forces). In contrast, an undeformable solid can directly transmit forces through its structure.*

6.1 Demonstration:

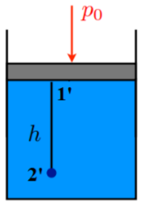
Consider an incompressible fluid with the density shown in the figure below.



by applying the hydraustatic principle between 1 and 2 :

$$p2 = p1 + \rho gh$$

Applying a force to the free surface results in an overpressure of p0.



The pressure at point 1' becomes: $p1' = p1 + p0$

Applying the principle of hydrostatics between points 1' and 2', we obtain the following equation:

$$p2' = p1' + \rho g h'$$

For an incompressible fluid, the density remains constant, and the fluid maintains a constant volume regardless of pressure changes. In this specific context the height difference between two points (h and h') is constant: $h = h'$

$$p2' = p1 + p0 + \rho g h$$

And at the end :

$$p2' = p1' + p0$$

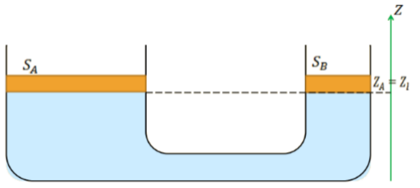
This demonstrates that a pressure change at point 1 is transmitted to point 2.

6.2 The Hydraulic Press:

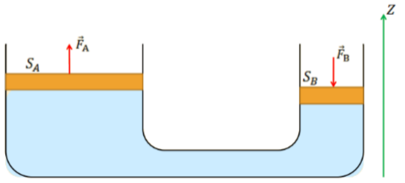
A mechanical device exploiting Pascal's principle, the hydraulic system enables mechanical force amplification by using an incompressible fluid (typically oil) in two cylinders of different diameters connected by a conduit; when a modest force is applied to the small-diameter piston, it generates a multiplied force on the larger-diameter piston, proportional to the ratio of surface areas, making possible power transmission and amplification in numerous industrial applications such as brakes, cylinders, hydraulic presses, and lifting systems.

Consider two cylindrical vessels with cross-sectional areas A and B, connected by a pipe filled with a liquid. Each vessel is sealed with a massless piston upon which masses M1 and M2 can be placed.

At equilibrium, the pressures beneath the pistons are equal, as points A and B lie on the same horizontal level.



When a small force is applied to the piston with area B, the piston with area A rises.



At equilibrium: $PA = PB$

$$PA = \frac{FA}{SA}$$

$$PB = \frac{FB}{SB}$$

$$\frac{FA}{SA} = \frac{FB}{SB} \quad \text{and}$$

$$FA = FB\frac{SA}{SB}$$

For example If the diameter ratio is ten, the force obtained FA is 100 times the initial force FB.

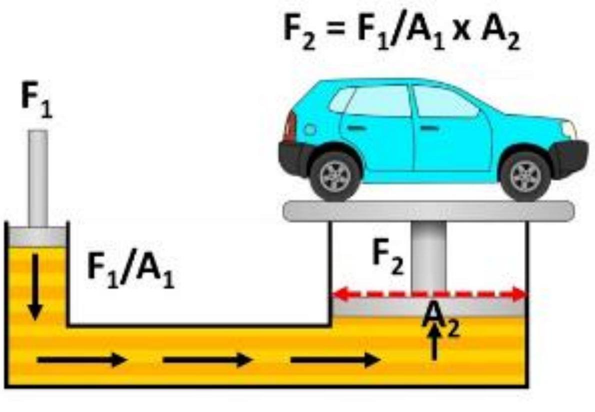


Figure 2. 7 Principle of a hydraulic press

6.3 Hydrostatic paradox

Regardless of the vessel's shape, if filled with the same liquid to the same height h, the bottom surface with area S experiences the same pressure force. This force equals the weight of a vertical column of fluid with base S and height h:

F = ρgh

This phenomenon is often referred to as the hydrostatic paradox: the upward force on the bottom of a container is independent of the shape of the container walls, assuming the walls are fixed.

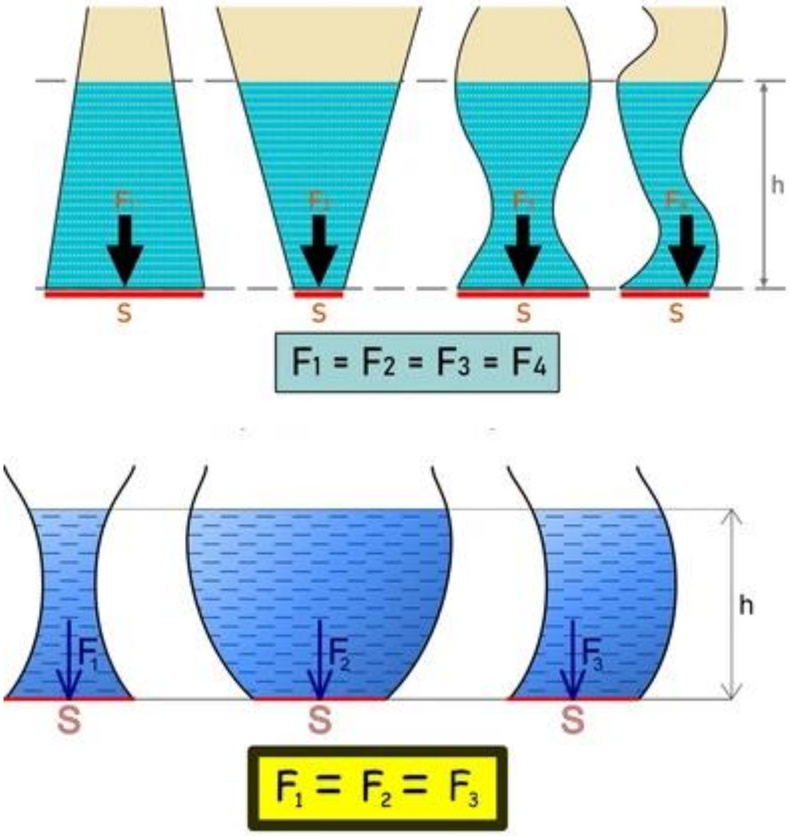


Figure 2. 8 Hydrostatic Paradox

7 Pressure Measurement Devices

Given the significance of pressure in fluid mechanics, it's unsurprising that a wide array of devices and techniques have been developed for its measurement.

7.1 The Barometer :

The primary function of a barometer is to measure atmospheric pressure. The first barometer was invented by the Italian scientist Evangelista Torricelli in 1644. He filled a meter-long glass tube, closed at one end, with mercury. Inverting the tube and submerging its open end in a mercury-filled bowl, he observed that the mercury level in the tube decreased, leaving a vacuum at the top. This experiment demonstrated the existence of atmospheric pressure.

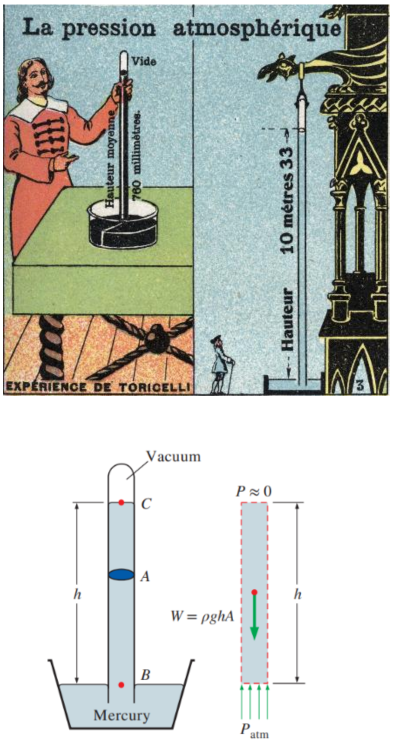


Figure 2. 9 The Torricelli experiment and the mercury barometer

$PB = PC + \rho gh$; $PC = Pvacuum = 0$; donc $PB = \rho gh = Patm$

$$h = 760mm$$

$$Patm = 101292 \text{ Pa}$$

At sea level: Patm=1 atm=1.0133.105 Pa, or 762 mm Hg 101325 Pa

Note: If water replaced mercury, the barometer would need a column about 10,36 meter high to measure standard atmospheric pressure (101325 Pa), as shown to scale in the Figure 2. 10.

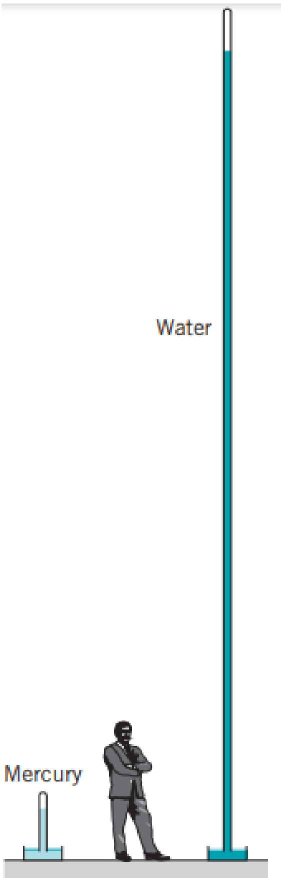
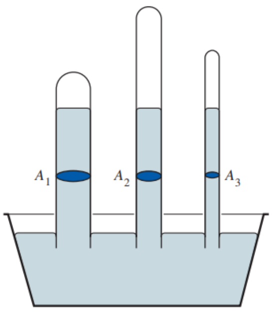


Figure 2. 10 Comparison of Water and Mercury Barometers

7.2 Mercury barometer.

Note : the cross-section of the barometer tube has no effect on mercury rise.



7.3 Manometry

Manometers are specialized scientific devices designed to measure fluid pressure by utilizing the principle of hydrostatic equilibrium, where a column of fluid is balanced against another fluid column to determine pressure differentials. These instruments are fundamentally categorized into two primary types: simple manometers, which measure absolute pressure at a single point, and differential manometers, which are used to determine pressure differences between two distinct points in a fluid system. By exploiting the relationship between fluid column height, density, and gravitational acceleration, manometers provide a precise and direct method of pressure measurement across various scientific and engineering applications.

7.4 Simple manometers

7.4.1 Piezometer Tube

The simplest manometer design consists of a vertical tube, open at the top, connected to the container where the pressure is to be measured, as shown in Figure

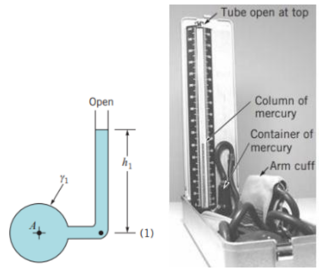


Figure 2. 11 Piezometer

We have:

$$PA = Patm + \rho gh$$

7.5 Manometers:

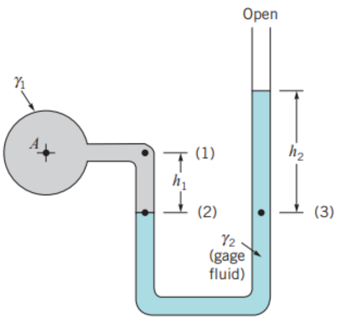
These devices consist of one or more curved tubes filled with one or more liquids of different densities.

Piezometers are commonly used to measure the pressure of liquids within tanks or pipes.

Piezometer is a sphygmomanometer, the classic device used to measure blood pressure.

7.5.1 U tube manometers

This device consists of a U-shaped tube containing a manometric liquid, such as mercury, water, alcohol, or oil. The tube is connected to a reservoir filled with a liquid or gas.



$$P2 = P3$$

$$P2 = PA + \gamma_1 h_1$$

$$P3 = Patm + \gamma_2 h_2$$

$$Patm + \gamma_2 h_2 = PA + \gamma_1 h_1$$

$$PA = Patm + \gamma_2 h_2 + \gamma_1 h_1$$

7.5.2 Differential U-tube manometer.

The device operates on a similar principle to a "U" gauge but instead measures the pressure differential between two distinct locations.

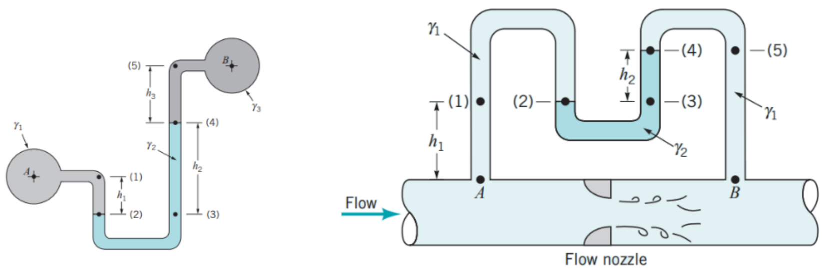


Figure 2. 12 Differential U-tube manometer.

If we take the figure on the left :

$$P2 = P3$$

$$P2 = PA + \gamma_1 h_1$$

$$P3 = PB + \gamma_2 h_2 + \gamma_3 h_3$$

$$PA - PB = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

7.5.3 Inclined differential manometer

Inclined-tube manometers offer precise measurements of small pressure differentials. One side of the manometer is positioned at an angle, and the differential reading is taken along this inclined tube.

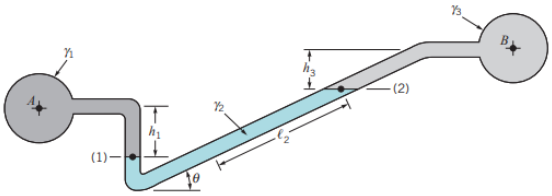


Figure 2. 13 Inclined-tube manometer.

$$P1 = PA + \gamma_1 h_1 = PB + \gamma_3 h_3 + \gamma_2 h_2$$

Or $\quad h_2 = l_2 sin(\theta)$

$P1 = PA + \gamma_1 h_1 = PB + \gamma_3 h_3 + \gamma_2 l_2 sin(\theta)$

$PA - PB = \gamma_3 h_3 + \gamma_2 l_2 sin(\theta) - \gamma_1 h_1$

Figure 2. 14 A simple U-tube manometer, with high pressure applied to the right side

7.6 Mechanical and Electronic Pressure Measuring Devices :

7.6.1 Mechanical Devices:

- Bourdon Gauge: Employs an elastic tube that deforms under pressure, converting this deformation into a mechanical movement.
- Aneroid Barometer: Measures absolute atmospheric pressure by detecting the deflection of an evacuated hollow element.

Figure 2. 15 Bourdon Gauge Aneroid Barometer

7.6.2 Electronic Devices:

- Pressure Transducers: Convert pressure into an electrical signal.
- Linear Variable Differential Transformer (LVDT): Transforms mechanical movement, induced by pressure, into a voltage signal.

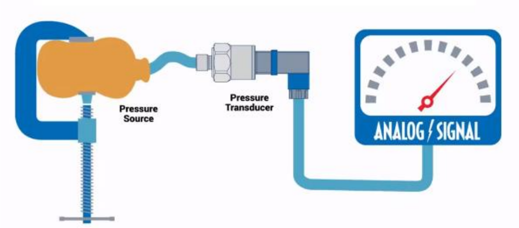


Figure 2. 16 Pressure Transducers

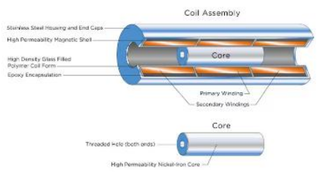


Figure 2. 17 Linear Variable Differential Transformer

Fundamental Principles:

- Conversion of pressure into a measurable mechanical deformation.
- Transformation of the deformation into an electrical signal.
- Capability to measure both static and dynamic pressures.
- Adaptability to various pressure ranges and types.

7.6.3 Advantages of Electronic Devices:

- Continuous measurement capability.
- Computer-based recording and data processing.
- High precision.
- Suitability for rapid pressure fluctuations.

7.7 Graphical representation of pressure

The graphical representation of pressure variation along a plane wall as a function of immersion depth is called a pressure distribution diagram or pressure diagram.

According to the fundamental equation of hydrostatics, the absolute pressure along a vertical plane wall varies linearly:

$$P = Patm + \gamma h$$

Since the effective pressure at point O, located at the free surface, is zero (the liquid surface is exposed to the atmosphere, i.e., atmospheric pressure in absolute terms), the gauge (effective) pressure along the wall varies linearly:

$$P_{gage} = \rho g h = \gamma h$$

Moreover, the diagram of hydrostatic gauge pressure is triangular, because the pressure depends only on the immersion depth h. In contrast, the absolute hydrostatic pressure diagram is trapezoidal, since at every point along the wall, the absolute pressure exceeds the gauge pressure by a value equal to atmospheric pressure.

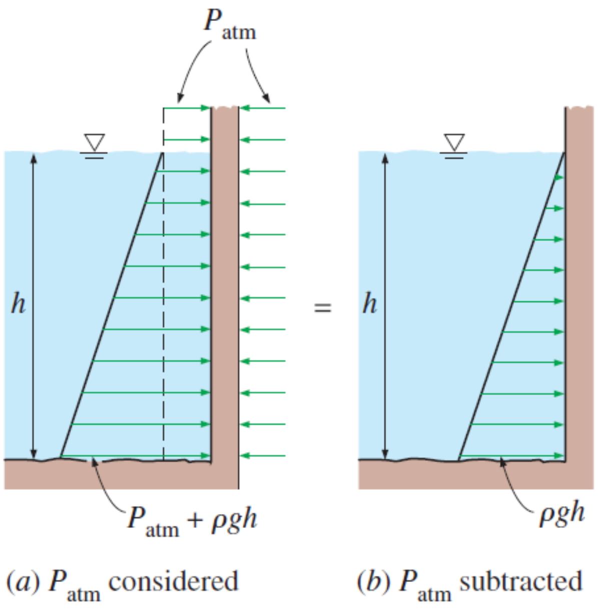


Figure 2. 18 Graphical representation of pressure : (a) Absolute pressure ; (b) gage pressure

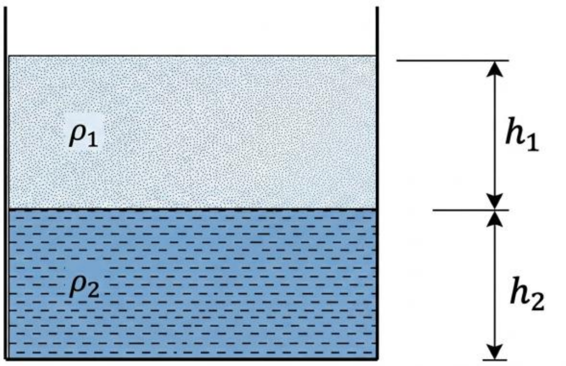
Exmple :

1. Construct the diagram of gauge pressure on the vertical wall of a container filled with two liquids, given that:

$\rho_1 = 0.8\rho_2$ and $h_1 = 1.5h_2$

With : $\rho_2 = 1250kg/m^3$ and $h_2 = 2m$

2. Determine the effective pressure at the interface between the two liquids and at the bottom of the tank.



Solution

1. Pressure distribution :

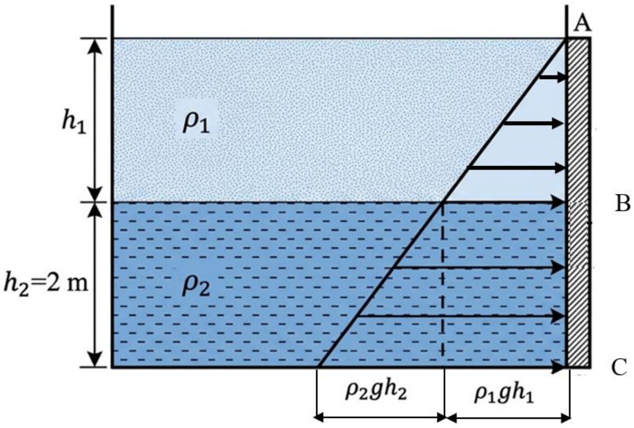
The hydrostatic gauge pressure diagram on the upper part **AB** is represented by a triangle whose base is:

$$P_{B,gauge} = \rho_1 g h_1$$

Now consider the lower part BC:

- At point B, a segment equal to $\rho_1 g h_1$ is drawn perpendicular to the wall BC, equal to the base of the upper diagram.
- At point C, the pressure is:

$$P_{C,gauge} = P_{B,gauge} + \rho_2 g h_2$$



2. Pressure calculations :

2.1 Effective pressure at the interface

$$P_{B,gauge} = 29430\ Pa$$

2.2 Effective pressure at the bottom

$$P_{C,gauge} = 53955\ Pa$$

8 Hydrostatic force on surfaces

In hydrotechnology, the practical interest lies in the force of the hydrostatic pressure of the liquid on the surface. Clearly, in order to design and build a hydraulic structure, such as a dam, we first need to determine the magnitude of the manometric pressure exerted by the liquid on the dam body, as well as its point of application. The parts of the workings subjected to hydrostatic pressure may be flat, curved or of any other shape.

In this subject, we focus on the most common shapes: the shape of flat surfaces (inclined, horizontal and vertical) and curved surfaces.

Figure 2. 19 Hydrostatic Force on dams Surfaces

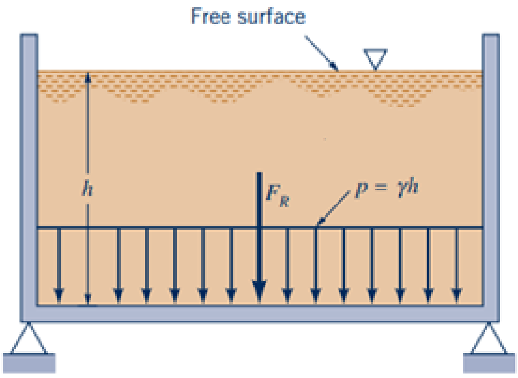
8.1 Hydrostatic force on a horizontal plane surface

Consider a tank open to the atmosphere, with a base area S, containing a liquid of density ρ to a depth h. When placed on a horizontal surface, the pressure exerted by the liquid is uniform across the entire base area.

$$P = \rho g h = \gamma h$$

$$F_R = \rho g h \, S = \gamma h S$$

S: The cross-sectional area of the tank bottom



;

Figure 2. 20 Hydrostatic force on a horizontal plane surface

The total force exerted by a fluid on a horizontal surface equals the product of the fluid column's weight (determined by height h) and the surface area of the wall. This force acts perpendicular to the surface, with its point of application located precisely at the geometric center (centroid) of the horizontal surface, regardless of the container's shape.

8.1.1 Hydrostatic force on an inclined plane surface:

Consider a Submerged Inclined Surface: An arbitrary-shaped flat surface of area A is immersed in a liquid, positioned at an angle θ relative to the fluid's free surface. Our objective is to determine both the resultant force acting on this surface and its point of application.

To analyze the forces involved, we consider an elementary surface area dA of the wall. The elementary force dF acting on this differential area results from the fluid's hydrostatic pressure distribution, which varies with depth. The total force can be found by integrating the pressure effects over the entire surface area, accounting for the variation in pressure with depth and the surface's orientation.

$dF = pdA = \rho ghdA = \rho gysin\theta dA$ with $h = ysin\theta$

The total force exerted by the fluid on the entire wall is equal to :

$$F_R = \int dF = \int_A \rho gysin\theta dA = \rho gsin\theta \int_A ydA$$

The quantity $\int ydA\ A$ is called the static moment of the A surface with respect to (ox).

$$\int_A ydA = y_c A$$

yC is the geometrical center or controid of the wall. Equation then becomes :

$$F_R = \rho gAy_c\, sin\theta = \rho gAh_c$$

Where:

$$h_c = y_c\, sin\theta$$

Where hC is the height of the liquid above the geometric center. So the resultant force is independent of the angle of inclination of the wall, it just depends on the density of the liquid, the surface area of the wall and the depth of the wall controid.

The point of application of FR is different from the geometric center of the wall. It is determined by calculating the moment of the resultant force with respect to (ox) and (oy).

moment of the resultant force :

$$F_R y_R = \int_A y dF = \int_A \rho g \, sin\theta \, y^2 dA$$

and, therefore, since $F_R = \rho g \, A \, y_c sin\theta$

$$y_R = \frac{\int_A y^2 dA}{A \, y_c}$$

$\int_A y^2 dA$ is the moment of inertia or quadratic moment with respect to (ox) noted IOX

$$I_{OX} = \int_A y^2 dA$$

$$y_R = \frac{I_{OX}}{A \, y_c}$$

The moment of inertia of surface A with respect to an axis parallel to (ox) and passing through the geometric center of the wall is denoted ICX. Applying Huyghens' theorem:

$$I_{OX} = I_{CX} + y_c^2 A$$

$$y_R = \frac{I_{CX}}{A \, y_c} + y_c$$

So the point of application of FR lies below the controid since :

$$\frac{I_{CX}}{A \, y_c} > 0$$

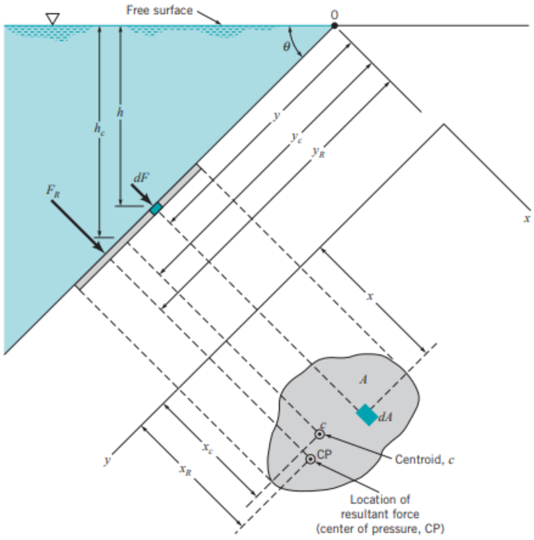


Figure 2. 21 hydrostatic force on an inclined plane surface

Centroidal coordinates and moments of inertia for some common areas are given in Figure 2. 22:

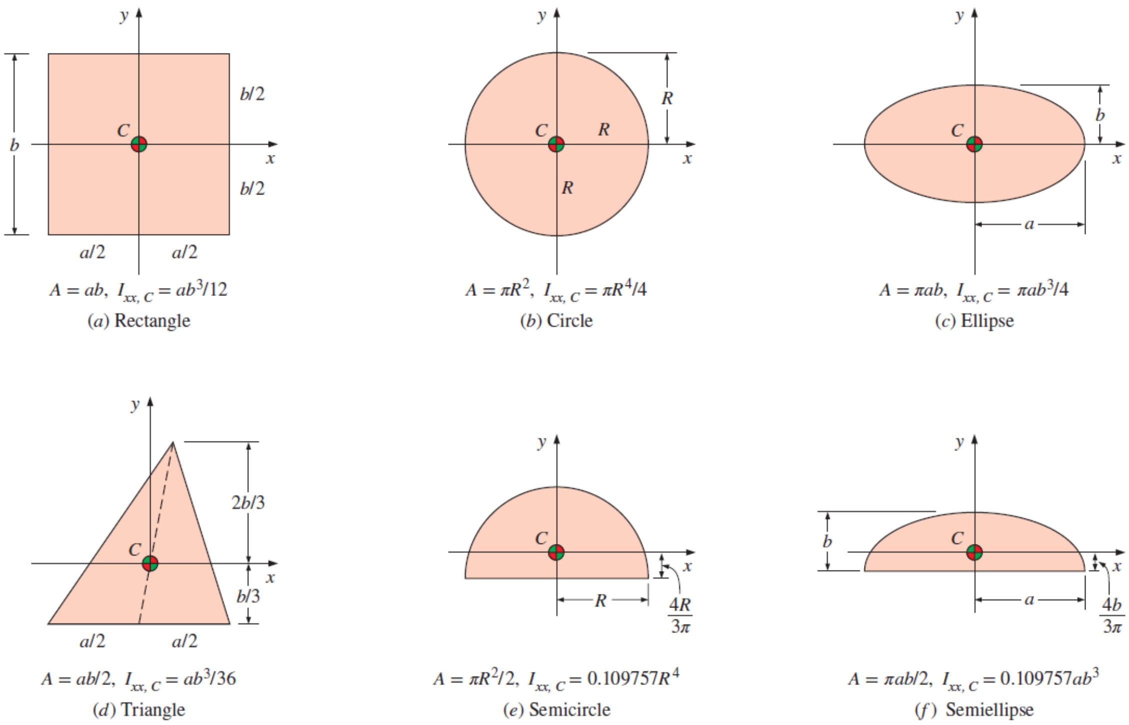


Figure 2. 22 Geometric properties of some common shapes

9 Hydrostatic Force on a Curved Surface

In many fluid mechanics applications, submerged surfaces are curved rather than flat (such as dams, tanks, and pipes). In this case, determining the resultant hydrostatic force becomes more complex.

Unlike plane surfaces, where the pressure direction is uniform, here the pressure always acts normal to the local surface, so its direction changes along the curved surface. This makes direct calculation difficult, and methods like the pressure prism are not very useful.

The simplest approach is to decompose the resultant force into two components:

- a horizontal component F_H
- a vertical component F_V

$$F_H = F_2$$

$$F_V = F_1 \pm W$$

The term $\pm W$ represents a vector sum, meaning that the magnitudes are added when both forces act in the same direction and subtracted when they act in opposite directions.

From this, we conclude that:

1. The horizontal component of the hydrostatic force acting on a curved surface is equal, both in magnitude and line of action, to the hydrostatic force exerted on the vertical projection of the surface.
2. The vertical component of the hydrostatic force acting on a curved surface is equal to the hydrostatic force on the horizontal projection, combined with the weight of the fluid above it—added if both act in the same direction, and subtracted if they act in opposite directions.

The magnitude of the resultant is :

$$F_R = \sqrt{(F_H)^2 + (F_V)^2}$$

and the tangent of the angle it makes with the horizontal is:

$$\tan \alpha = \frac{F_V}{F_H}$$

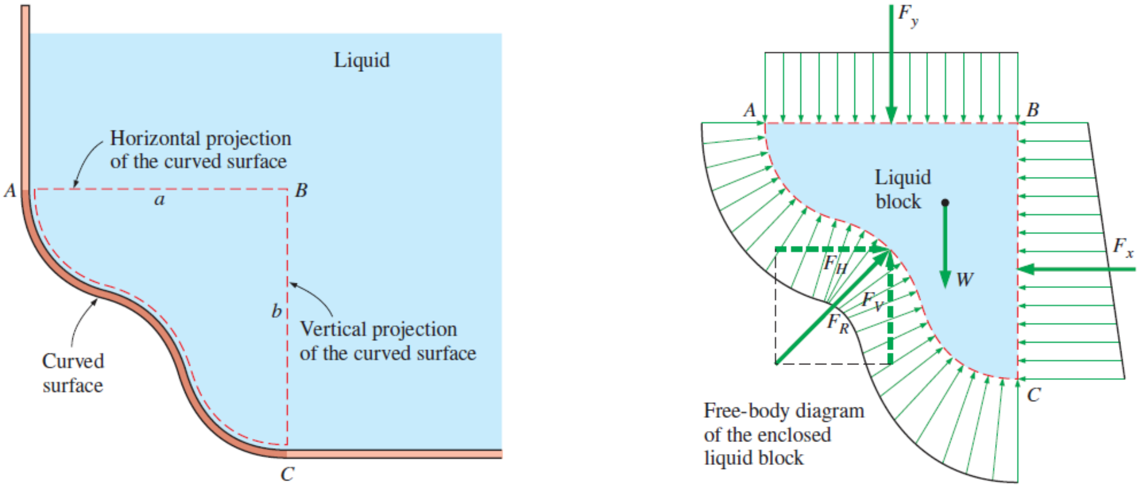
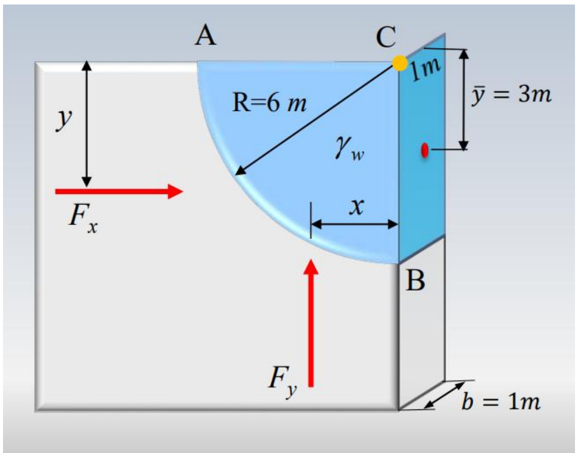


Figure 2. 23 Determination of the hydrostatic force acting on a submerged curved surface.

The center of pressure on a curved surface is the point of application of the resultant hydrostatic force, whose position is determined using moment equilibrium. In general, it is obtained by combining the horizontal and vertical components of the hydrostatic force. The horizontal component acts through the center of pressure of the vertical projection, while the vertical component acts through the centroid of the fluid volume above the surface.

Example1:

Determine the components of the hydrostatic force and their points of application in the following case.



The horizontal component is given:

$$F_X = \gamma \bar{y} A_{BC} = 9810x3x(6X1)$$

$$F_X = 176.6\ kN$$

$$F_Y = \gamma V_{ABC} = 9810x\left(\frac{\pi 6^2}{4}\right)x1$$

$$F_Y = 277.4\ kN$$

The center of pressure in the x-direction coincides with the centroid:

$$x_p = \frac{4R}{3\pi} = \frac{4x6}{3\pi} = 2.55m$$

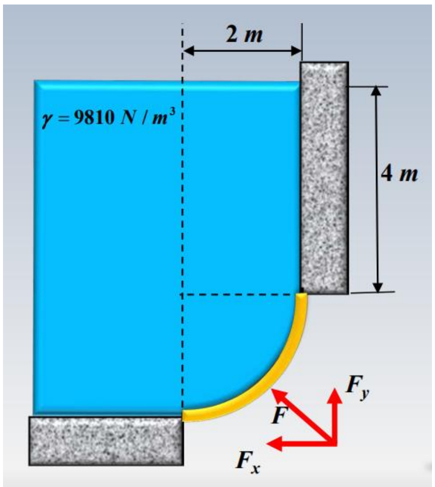
the center of pressure in the y-direction is given as follows:

$$y_p = \bar{y} + \frac{I_{CX}}{A\ y_c} = 3 + \frac{(1x6^3)/12}{3x6x1} = 1m$$

$$\bar{y} = 3m$$

Example 2:

Determine the horizontal and vertical components F_X and F_Y of the reaction force, their points of application, and finally the resultant force. the thickness of 1 m.



Calculation of F_X

$$F_X = \gamma \bar{h} A = 9810x3x(6X1)$$

$\bar{h} = 4 + 1$ and $A = 2x1$

$$F_X = 9810x5x2$$

$$F_X = 98.1\ kN$$

Calculation of F_Y

$$F_Y = the\ weight\ of\ the\ fluid\ above\ the\ gate$$

$$F_Y = F_V + W$$

$$F_V = 9810x(4x2x1) = 78.5kN$$

$$W = 9810x(0.25x\pi 2^2 x1) = 30.8\ kN$$

$$F_Y = 30.8 + 78.5 = 109.3\ kN$$

Point of application of F_X:

$$y_p = \bar{y} + \frac{I_{CX}}{A\ y_c}$$

$$\bar{y} = 4 + 1 = 5\ m$$

$$I_{CX} = \frac{bh^3}{12}$$

$$y_p = 5 + \frac{\frac{1x2^3}{12}}{2x1x5}$$

$$y_p = 0.067\ m$$

Point of application of F_Y

We are looking for x_p using the sum of moments, we obtain:

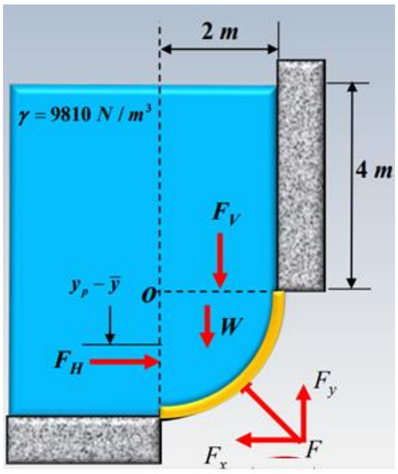
$$x_p F_Y = F_V.1 + W.\overline{x_W}$$

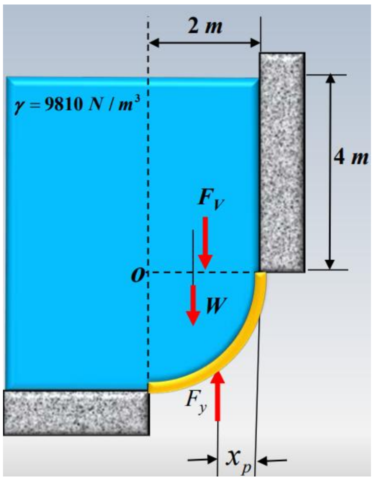
With: $\overline{x_W}$: Center of pressure in the x-direction = centroid

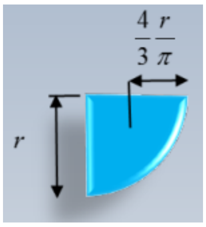
$$\overline{x_W} = \frac{4.R}{3\pi} = 0.849m$$

$$x_p = \frac{F_V.1 + W.\overline{x_W}}{F_Y} = \frac{78.5x1 + 30.8x0.849}{109.3}$$

$$x_p = 0.957\ m$$







The resultant hydrostatic force is defined by its magnitude and the tangent of the angle it makes with the horizontal:

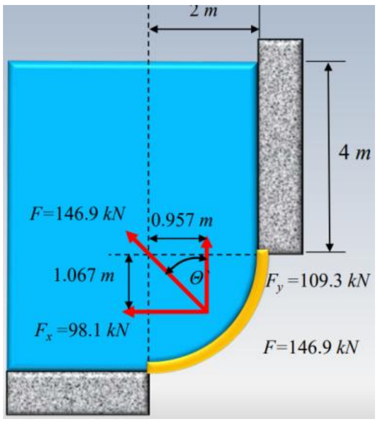
$$F = \sqrt{F_X^2 + F_Y^2}$$

$$F = 146.9\ kN$$

$$\theta = tan^{-1}\frac{F_Y}{F_X} = tan^{-1}\frac{98.1}{109.3}$$

$$\theta = 42°$$

The representation of the hydrostatic force and its point of application is illustrated in the figure below.

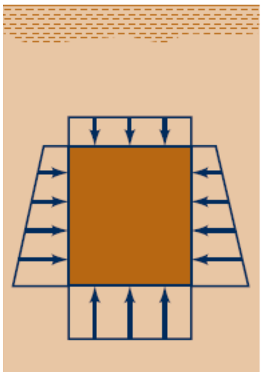


10 Buoyancy, Flotation, and Stability : Archimedes' Principle

10.1 Definition and origin of buoyancy

When a solid body is immersed in a fluid — either fully or partially — it experiences a net upward vertical force exerted by the fluid. This force is called the buoyant force. It arises from the fact that fluid pressure increases with depth: the pressure forces acting on the bottom face of the body are greater than those acting on the top face.

Consider a body of volume V submerged in a fluid of specific weight γ. We enclose it within a fictitious parallelepiped and analyze the forces acting on the fluid volume with the body removed:



- F_1 and F_2: pressure forces on the horizontal (top and bottom) faces
- F_3 and F_4: pressure forces on the vertical faces — they cancel by symmetry
- $\mathcal{W}$: weight of the fluid in the shaded region (parallelepiped minus body)
- F_B: the force exerted by the body on the fluid (reaction force)

Applying equilibrium in the vertical z-direction:

$$\mathbf{F_B = F_2 - F_1 - \mathcal{W}}$$

Assuming constant specific weight γ throughout the fluid:

$$F_2 - F_1 = \gamma(h_2 - h_1)A$$

where A is the horizontal cross-sectional area. Substituting and simplifying gives the fundamental result:

$$\mathbf{F_B} = \gamma V$$

10.2 Archimedes' Principle

The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

This result, known since antiquity through the work of Archimedes (287–212 B.C.), can be derived rigorously from fluid statics.

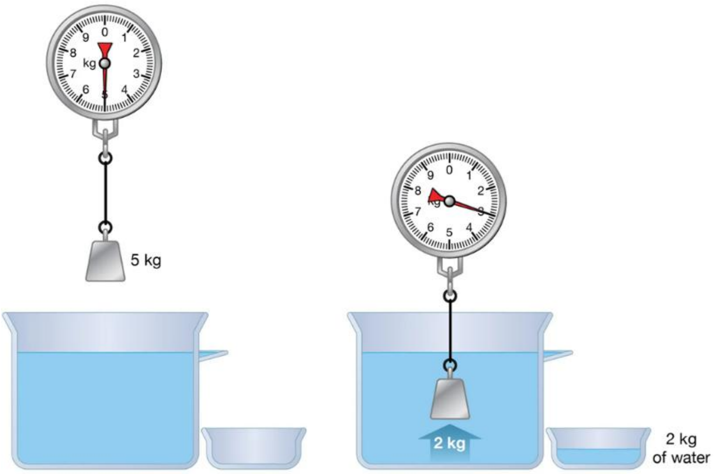


Figure 2. 24 Archimedes' principle of buoyancy

In Figure 2.24, a 5 kg object immersed in water is subjected to an upward buoyant force of 2 kg, which is equal to the weight of the displaced water. This buoyant force reduces the object's apparent weight by 2 kg, decreasing it from 5 kg to 3 kg.

10.3 Equilibrium and Flotability

The behavior of a body in a fluid depends on the ratio of its average density (ρ) to the fluid density (ρ_f):

1. **Sinking body:** If $\rho < \rho_f$, the weight is greater than the buoyant force (W > F_B). The object sinks.
2. **Neutrally buoyant body:** If $\rho = \rho_f$, the object remains at rest at any location within the fluid.
3. **Floating body:** If $\rho > \rho_f$, the object rises to the surface. At equilibrium, only a fraction of the volume is submerged V_{sub}, such that the weight of the entire body equals the weight of the fluid displaced by the submerged portion.

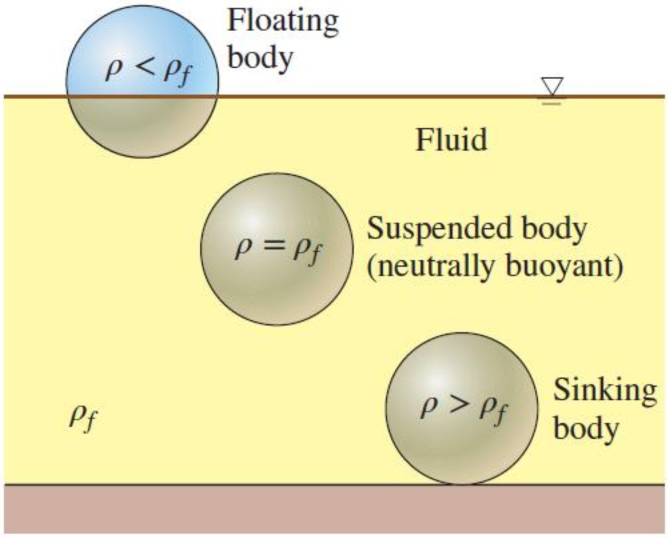


Figure 2. 25 Flotation and immersion conditions in fluids

10.4 The Centre of Buoyancy

The centre of buoyancy is the centroid of the displaced fluid volume, and the point through which the buoyant force acts vertically upward. Unlike the centre of gravity — which reflects the body's mass distribution and acts downward — the centre of buoyancy depends solely on the geometry of the submerged portion.

Figure 2. 1 illustrates that these two points do not necessarily coincide:

- When partially submerged, the asymmetric displaced volume creates a separation between the two centres, potentially generating a rotational moment.
- When fully submerged with uniform density, the displaced volume becomes symmetric and both centres tend to coincide.

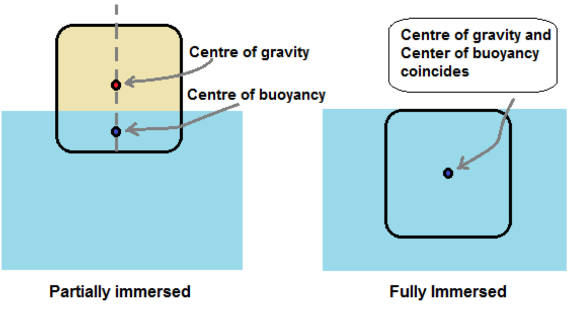


Figure 2. 26 The difference between the centers of buoyancy and gravity

10.5 Stability of Immersed and Floating Bodies

10.5.1 Concepts of Stability

Stability can be understood through the "ball on a floor" analogy (Figure 2. 27):

- ✓ Stable: Returns to equilibrium after a disturbance.
- ✓ Neutrally stable: Remains in the new position.
- ✓ Unstable: Moves away from the equilibrium point.

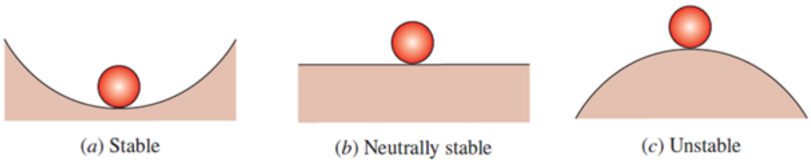


Figure 2. 27 Understanding Stability Through the Example of a Ball

10.5.2 Stability of Immersed Bodies

It is determined by the relative locations of the center of gravity (G) and the center of buoyancy (B):

- ✓ Stable: G is below B. A disturbance produces a restoring moment.
- ✓ Neutrally stable: G and B coincide.
- ✓ Unstable: G is above B. Any disturbance causes the body to overturn.

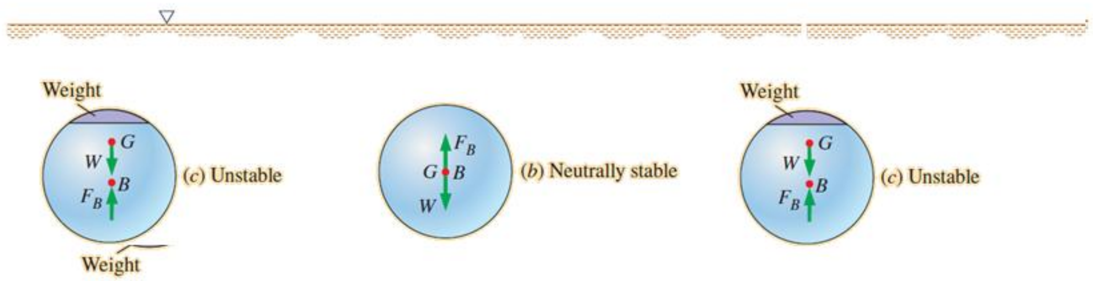


Figure 2. 28 Stability of a completely immersed body

10.5.3 Metacenter and Stability of Floating Bodies

The metacenter (M) is the point where the vertical line through the shifted centre of buoyancy (B') intersects the original vertical axis of the body after a small angular displacement. It is a key concept in determining whether a floating body will return to equilibrium or capsize.

The metacentric height (GM) is the vertical distance between the centre of gravity (G) and the metacenter (M), and serves as a direct measure of stability:

- Stable equilibrium — M is above G (GM > 0): the buoyant force generates a restoring moment that brings the body back to its upright position.
- Unstable equilibrium — M is below G (GM < 0): the buoyant force amplifies the tilt, producing an overturning moment that causes capsizing.
- Neutral equilibrium — M coincides with G (GM = 0): the body remains in whatever tilted position it is placed, neither recovering nor capsizing.

A floating body is also inherently stable when it is bottom-heavy, meaning G lies directly below B, regardless of the metacenter position. As illustrated in Figure 2. 29, typical metacentric height values range from 0.3–0.7 m for cruise ships, 0.9–1.5 m for sailboats, and 0.75–1.3 m for warships.

In practice, a larger GM improves stability but can lead to stiff, uncomfortable rolling motion. Naval architects must therefore strike a balance between adequate stability and acceptable dynamic behaviour when designing ships and floating structures.

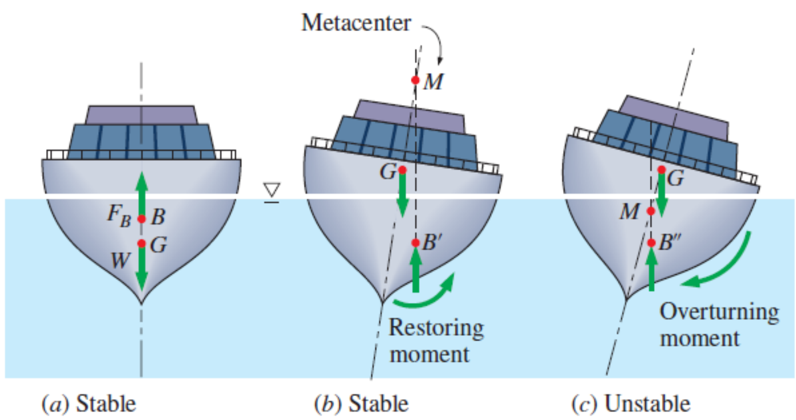


Figure 2. 29 Stability conditions of a floating body

PRACTICAL EXERCISES

Exercise N°1

Express a pressure of 155 kPa (measured) as absolute pressure. Local atmospheric pressure is 98 kPa (abs)

Solution N°1 :

By definition, we have :

$$Pabs = Prel + Patm$$

Note that relative pressure is sometimes referred to as measured pressure.

$$Pabs = =155 + 98 = 253 \text{ kpa}$$

Exercise N°2 :

The standard atmospheric pressure is 101325 Pa. Calculate its equivalent in height of mercury.

Solution N°2:

Atmospheric pressure can be expressed as :

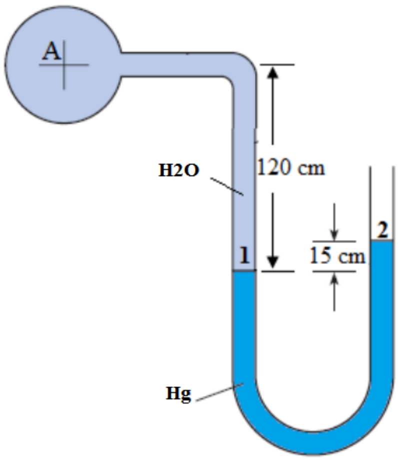
$$Patm = \rho_{Hg}.g.h = 101325\ Pa$$

Therefore:

$$h = \frac{Patm}{\rho_{Hg}.g} = 0.760\ mHg = 760mmHg$$

Exercise N°3:

Calculate the gauge pressure at A in kPa due to the difference in level of mercury, density 13.6, in the U-shaped manometer shown in the figure below:



Solution N°3:

Let's apply the fundamental law of hydrostatics (FLH) between points A and 1, and then between points 1 and 2.

$$P_1 = P_A + \rho g(0.12)$$

$$P_1 = P_2 + \rho_{Hg} g(0.15) = P_{atm} + \rho_{Hg} g(0.15)$$

Since we are calculating gauge pressure, we subtract Patm and find :

$$P_1 = \rho_{Hg} g(0.15) = 13600x9.81x0.15 = 20012.4\ Pa$$

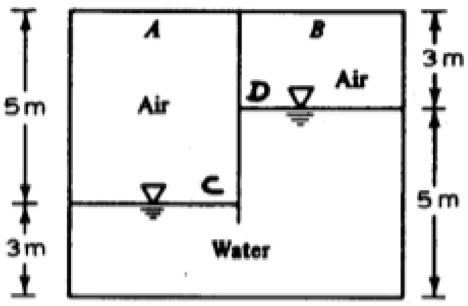
$$P_A = P_1 - \rho g(0.12) = 8240.4\ Pa = 8.24\ kPa$$

Exercise N°4:

In the figure below, a closed tank is maintained at 20°C. If the pressure at point A is 98 Pa, determine:

1. The absolute pressure at point B
2. Calculate the percentage error that would result from ignoring the specific weight of air

Note: specific weight of air 0.0118 kN/m^3 , specific weight of water at 20°C = 9.79kN/m^3



Solution N°4:

1- Assuming air behaves as an incompressible fluid, we can apply the fundamental law of hydrostatics, which gives us:

between C and A :

$$PC = PA + \rho_{air} g h_{A-C}$$

between C and B:

$$PC = PB + \rho_{air} g h_{D-B} + \rho_{water} g h_{D-C}$$

Therefore:

$$PA + \rho_{air} g h_{D-B} + \rho_{water} g h_{D-C} = PB + \rho_{air} g h_{A-C}$$

$$PB = PA + \rho_{air} g h_{A-C} - \rho_{air} g h_{D-B} - \rho_{water} g h_{D-C}$$

$$PB = 78.444 KPa$$

2- By neglecting the specific weight of air:

between C and A :

$$PC = PA$$

between C and B:

$$PC = PB + \rho_{water} g h_{D-C}$$

Therefore:

$$PB = PA - \rho_{water} g h_{D-C}$$

$$PB = 98 - (9.790)(5-3) = 78.420\ kPa$$

$$error = \frac{(78.444 - 78.420)}{78.444} = 0.00031$$

Which is given as a percentage: 0.031%

Exercice N°5:

An aircraft's barometer indicates an outside pressure of 685 mm of mercury. If the barometric pressure at sea level is 760 mm of mercury and the air density is 1.21 kg/m3, how high is the plane flying?

Solution N°5:

Using the fundamental law of hydrostatic and by considering air as an incompressible fluid:

Let us convert the pressures to Pascal (Pa):

$$P1 = \rho_{Hg}.0.760 = 760 \times 133.322 = 101324.72 \text{ Pa}$$

$P2 = 685 \text{ mm Hg} = 685 \times 133.322 = 91325.57 \text{ Pa=}$

Using the fundamental law of hydrostatics:

$$P1 = P2 + \rho g h$$

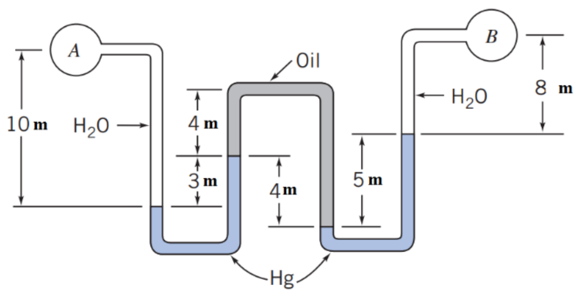
Where h is the height we are looking for

$$h = \frac{P1 - P2}{\rho g}$$

h = 842 meters

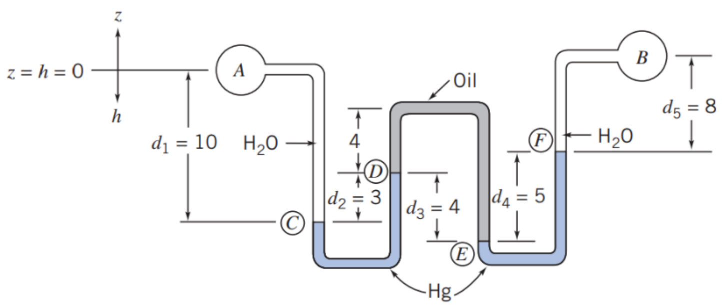
Exercise N° 6:

Water flows through pipes A and B. The manometer contains lubricating oil in the upper part of the inverted U-tube and mercury in the lower part. Determine the pressure difference, pA - pB.



Solution N°6 :

Applying the fundamental relation of hydrostatics, starting at point A, we get :



$$PC - PA = \rho_{H2O} g d_1$$

$$PD - PC = -\rho_{Hg} g d_2$$

$$PE - PD = \rho_{Oil} g d_3$$

$$PF - PE = -\rho_{Hg} g d_4$$

$$PB - PF = -\rho_{H2O} g d_5$$

Let's combine these equations:

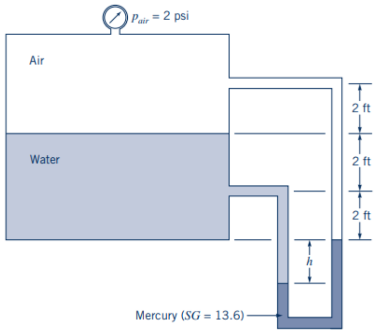
$$PB - PA = \rho_{H2O} g d_1 - \rho_{Hg} g d_2 + \rho_{Oil} g d_3 - \rho_{Hg} g d_4 - \rho_{H2O} g d_5$$

$$PB - PA = -504.2 \text{ kPa}$$

The negative value indicates that the pressure at point A is lower than the pressure at point B.

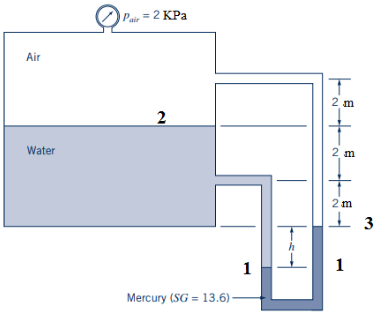
Exercise N°7:

A U-tube mercury manometer is connected to a closed pressurized tank as shown in the figure. If the air pressure is 2 kPa, determine the differential height reading, h. Assume the specific weight of air is negligible.



Solution N°7:

At points 1 and 1 (same level), the pressures must be equal: P1(left) = P1(right)



P1(left) :

$$P1 = P2 + \rho_{water} g h_{1-2}$$

$h_{1-2} = h + 2 + 2 = h + 4$ and $P2 = P_{air}$

$$P1 = P_{air} + \rho_{water} g(h + 4)$$

P1(right):

$$P1 = P3 + \rho_{Hg} g h_{1-3}$$

$P3 = P_{air}$ and $h_{1-3} = h$, so:

$$P1 = P_{air} + \rho_{Hg} g h$$

$$P_{air} + \rho_{water} g(h + 4) = P_{air} + \rho_{Hg} g h$$

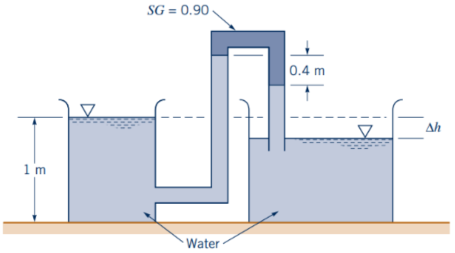
$$(\rho_{Hg} - \rho_{water}) h = \rho_{water} 4$$

$$h = \frac{\rho_{water} 4}{(\rho_{Hg} - \rho_{water})}$$

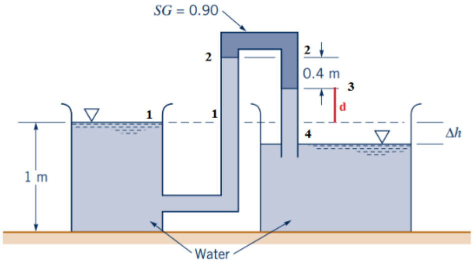
$$h = \frac{1000.4}{(13600 - 1000)} = 0.36m$$

Exercise N°8:

Determine the elevation difference, Δh, between the water levels in the two open tanks shown in Figure below.



Solution N°8:



$P1 = Patm = P2 + \rho_{H2O} g h_{1-2}$ same level in the same liquid

$$P4 = Patm = P2 + \rho_{H2O} g h_{4-3} + \rho_{Hg} g h_{3-2}$$

We have:

$h_{1-2} = d + 0.4$ and $h_{4-3} = \Delta h + d$

$$P2 = \rho_{H2O} g h_{1-2} - Patm$$

$$P2 = Patm - \rho_{H2O} g h_{4-3} - \rho_{Hg} g h_{3-2}$$

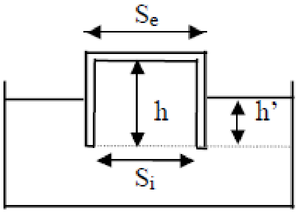
$$Patm - \rho_{H2O} g(\Delta h + d) - \rho_{Hg} g 0.4 = \rho_{H2O} g(d + 0.4\ ) - Patm$$

$$\rho_{H2O} g(\Delta h) = \rho_{H2O} g(0.4\ ) + \rho_{Hg} g 0.4$$

$$(\Delta h) = 0.4 \frac{(\rho_{H2O} + \rho_{Hg})}{\rho_{H2O}} = 5.04m$$

Exercise N°9:

Consider a cylindrical glass of mass m, internal height h, internal cross-sectional area Si, and external cross-sectional area Se. The glass is filled completely with water and then inverted and submerged to a depth h' in a water tank.

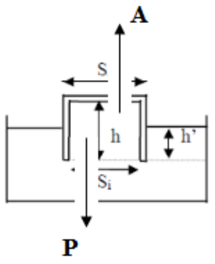


What force must be applied by the operator to keep the glass in equilibrium?

Solution N°9 :

Applying Archimedes' theorem, we get:

$$\sum \vec{F} = \vec{0}$$



By projection along the z axis :

$P - A = 0$ So $P = A$

$$P = (m + \rho_{H20} Si.h)g$$

$A = \rho_{H20}.V_{im}.g$ and therefore :

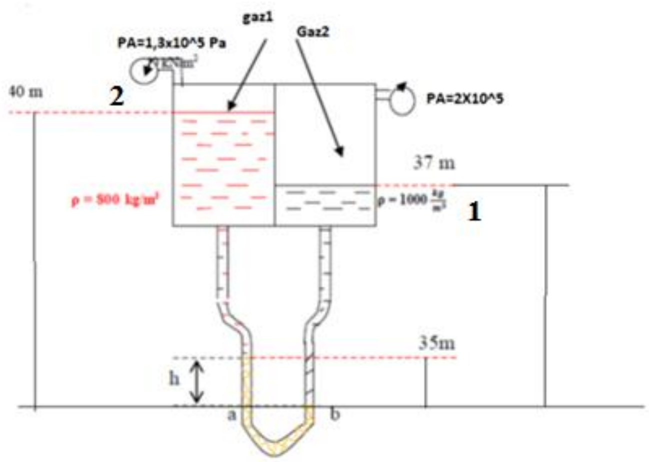
$$A = \rho_{H20}.(Se.h').g$$

$$(m + \rho_{H20} Si.h)g = \rho_{H20}.(Se.h').g$$

$$h' = \frac{(m + \rho_{H20} Si.h)}{\rho_{H20}.(Se)}$$

Exercise N°10 :

Calculate the height difference h of the U-shaped mercury manometer?



Solution N°10:

We have:

$$Pa = Pb$$

$$Pa = P2 + \rho_{oil} g(40 - 35) + \rho_{Hg} gh$$

$$Pb = P1 + \rho_{H20} g37$$

$$P2 + \rho_{oil} g(40 - 35) + \rho_{Hg} gh = P1 + \rho_{H20} g37$$

$$h = \frac{P1 + \rho_{H20} g37 - P2 - \rho_{oil} g(40 - 35)}{g\rho_{Hg}}$$

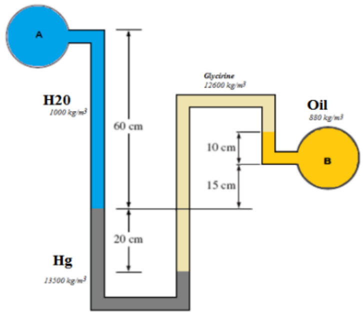
on a : $P2 = 1.3.10^5 Pa$ and $P1 = 2.10^5 Pa$

$$h = \frac{2.10^5 + 1000.9.81.37 - 1.3.10^5 - 800.9.81(40 - 35)}{9.81.13600}$$

$$h = 0.24m$$

Exercise N°11:

The pressure difference between the oil tube and the water tube is measured by a double manometer, as shown in the figure opposite. For the heights and densities of the fluids involved, calculate the pressure difference Δp=pB-pA



Solution N°11:

We consider two points (1) and (2) at equal pressure, as illustrated in the figure. We can write :

$P_1 = P_2$

$p_1 = \rho_{Hg}.g.h_{Hg} + \rho_{H2O}.g.h_{H2O} + p_A$

$p_2 = \rho_{Glyc.}.g.h_{Glyc.} + \rho_{Huile}.g.h_{Huile} + p_B$

$$P_1 = P_2$$

$$P_1 = \rho_{Hg}.g.h_{Hg} + \rho_{H2O}.g.h_{H2O} + P_A$$

$$P_2 = \rho_{Glyc}.g.h_{Glyc} + \rho_{oil}.g.h_{oil} + P_B$$

Then:

$$\rho_{Hg}.g.h_{Hg} + \rho_{H2O}.g.h_{H2O} + P_A = \rho_{Glyc}.g.h_{Glyc} + \rho_{oil}.g.h_{oil} + P_B$$

$$P_B - P_A = \rho_{Hg}.g.h_{Hg} + \rho_{H2O}.g.h_{H2O} - (\rho_{Glyc}.g.h_{Glyc} + \rho_{oil}.g.h_{oil})$$

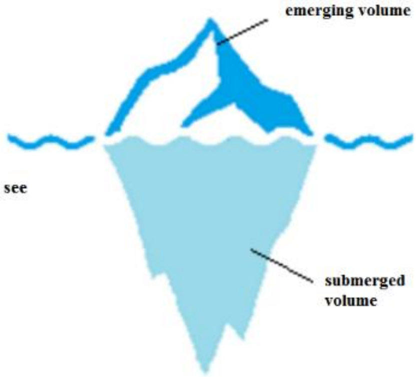
$$P_B - P_A = 13600x9.81x0.2 + 1000x9.81x0.6 - (12600x9.81x(0.2 + 0.15 + 0.1) + 880x9.81x0.1)$$

$$P_B - P_A = -23916.8\,Pa$$

Exercise N°12:

An iceberg has a total volume VTotale = 600 m3 . Its density is ρGlace = 910 $kg.m^{-3}$, that of seawater is ρH2O = 1024 $kg.m^{-3}$.

1- Draw a diagram of the floating iceberg and specify the forces to which it is subjected when in equilibrium.
2- Find a relationship between the emerged volume ($V_{emerging}$), the total volume (V_{total}) and the densities (ρ_{ice}, ρ_{water}).
3- Calculate the volume V_{total} and the mass m of the iceberg.
4- The question is: does the iceberg float on water? Given: g=9.81 m/s².



Solution N°12:

The weight of the iceberg (ice)

W=m.g=ρ.V_{total}.g=910x600x9.81=5.35x10^6 N

The Archimedean thrust exerted by the water on the iceberg :

$P_A = \rho_{eau}.V_{Immergé}.g = V_{Immergé}.1024.9, 81$

$P_A = \rho_{water}.V_{Submerged}.g = V_{Submerged}x1024.81$

The iceberg is in equilibrium under the action of its weight and buoyancy. These two opposing forces have the same norm (equilibrium), so we write (1) = (2):

$5.35x10^6=9.8x1024xV_{Submerged}$

$V_{Submerged}=546\ m^2$

With:

The relationship between volumes ($V_{emerged}$, V_{total}) and densities. Writing the equality between (1) and (2) at equilibrium :

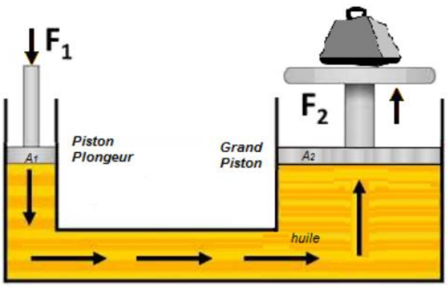
$V_{Submerged}x\rho_{water} = V_{total}x\rho_{ice}$

$V_{Submerged}/V_{total} = \rho_{ice}\ /\rho_{water}$

The 89% of the iceberg's volume is underwater. So the iceberg floats.

Exercise N°13:

A hydraulic press has a 30 cm-diameter piston and a 4.5 cm-diameter plunger. Find the weight that can be lifted by the hydraulic press when the force applied to the plunger piston is 500 N.



Solution N°13:

The diameter of the large piston is D = 0.3 m; the diameter of the plunger piston is d = 0.045 m. The force F1 applied to the plunger piston is 500 N. The question is to find the weight that can be lifted on the other side of the press.

We have so:

$$A_1 = \frac{\pi D^2}{4} = \frac{3.14x0.3^2}{4} = 0.07068\ m^2$$

$$A_2 = \frac{\pi d^2}{4} = \frac{3.14x0.045^2}{4} = 0.00159\ m^2$$

When F1 is in equilibrium with F2 (the weight to be lifted), we have :

$$P_1 = P_2$$

$$P_1 = \frac{F_1}{A_1} = P_2 = \frac{F_2}{A_2}$$

Where:

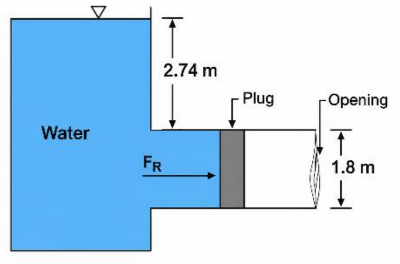
$\frac{F_1}{A_1} = \frac{F_2}{A_2}$

which gives:

$$F_2 = F_1 \frac{A_2}{A_1} = 500 \frac{0.07068}{0.00159} = 22226.41\ N$$

Exercise N°14:

A large open tank holds water and is connected to a 1.8 m diameter pipe, as shown in the adjacent figure. A circular plug is used to seal the pipe. Determine the magnitude, direction and location of the water force on the plug.



Solution N°14:

We have:

$F_R = \rho . g . h_C . A$ with $h_C = 3.64\ m$

$$A = \frac{\pi D^2}{4} = \frac{3.14x1.8^2}{4} = 2.54 m^2$$

Therefore:

$$F_R = 1000x9.81x3.64x2.54 = 90820.74\ N$$

$$y_R = \frac{I_{XC}}{y_C} + y_C$$

With:

$$I_{XC} = \frac{\pi D^2}{4} = 19.38\, m^4$$

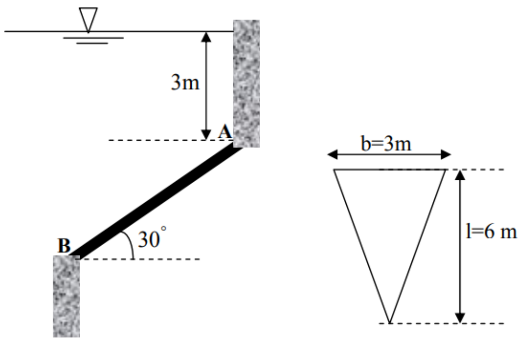
And finally:

$$y_R = \frac{\frac{\pi D^2}{4}}{3.64 x \pi x 1.8^2} + 3.64 = 3.93m$$

The force of 90820.74 acts 3.93 m below the free surface of the water and is perpendicular to the surface of the cork as shown in Fig.

Exercise N°15:

The diagram shows a triangular AB valve holding back a level of water and immersed at a depth of 3 m. Calculate the pressure force F exerted on this valve and its center of thrust: 1. if the vertex of the triangle is at B 2. if the vertex of the triangle is at A .



Solution N°15:

1- The vertex of the triangle is at B:

$$F = \rho g h_G S = 10^3 x 9.81 \left(\frac{1}{3} 6 \sin(30°) + 3\right) \frac{18}{2} = 353.16\, kN$$

$$y_p = y_G + \frac{I_{XG}}{y_G S}$$

$y_G = \frac{h_G}{sin30°} = 8m$ and $I_{XG} = \frac{36^3}{36} = 18\, m^4$

Therefore:

$$y_p = 8 + \frac{18}{8x9} = 8.25\, m$$

2- The vertex of the triangle is at A

$$F = \rho g h_G S = 10^3 x 9.81 \left(\frac{2}{3} 6\sin(30°) + 3\right)\frac{18}{2} = 441.45\ kN$$

$$y_p = y_G + \frac{I_{XG}}{y_G S}$$

$y_G = \frac{h_G}{sin30°} = 10\ m$ and $I_{XG} = \frac{36^3}{36} = 18\ m^4$

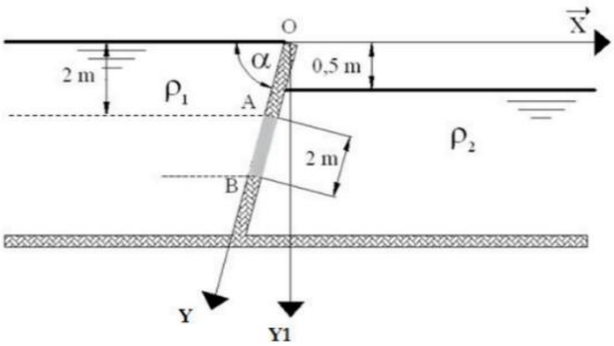
So:

$$y_p = 10 + \frac{18}{10x9} = 8.2\ m$$

Exercise N°16:

The figure shows a tank delimiting two liquid media of different densities, whose levels are offset by 0.5m. Consider a door AB of square cross-section inclined at an angle of α to the horizontal with a side length of L=2m. Given ρ2=850 Kg/m^3 , ρ 1= 1000 Kg/m^3 , α = 75°. We take g=10 m/s^2 . $I_{xg} = \frac{L^4}{12}$

1- Calculate the pressure force F1 exerted by liquid 1 on the square surface.
2- Specify the position of the center of thrust of force F1 on the Y axis and on the Y1 axis.
3- Calculate the pressure force F2 exerted by liquid 2 on the square surface.
4- Specify the position of the center of thrust of force F2 on the Y axis and on the Y1 axis.
5- Knowing that the joint is at point A, determine the force F to be applied at B to balance the door. Indicate the direction of force F.



Solution N°16:

1- Force F1:

$$F_1 = \rho_1 g h_{G1} S$$

$$F_1 = 10^4(2 + 1\,\sin\,(75°))4$$

$$F_1 = 118637.03\;N$$

2- F1 force center of thrust:
Y axe:

$$y_{p1} = y_{G1} + \frac{I_{xg}}{y_{G1}S}$$

$$y_{G1} = \frac{h_{G1}}{sin75°} = 3.07\;m$$

$$I_{xg} = \frac{L^4}{12} = 1.33\;m^4$$

$$y_{p1} = 3.07 + \frac{1.33}{3.07x4} = 3.178\;m$$

Y1axe:

$$h_{p1} = y_{p1} sin75° = 3.07m$$

3- Force F2:

$$F_2 = \rho_2 g h_{G2} S$$

$$F_2 = 850x10(2 + 1\,\sin(75°) - 0.5)4$$

$$F_1 = 83841.478\;N$$

4- Center of thrust of force F2
Y axe:

$$y_{p2} = y_{G2} + \frac{I_{xg}}{y_{G2}S}$$

$$y_{G2} = \frac{h_{G2}}{sin75°} = 2.55\;m$$

$$I_{xg} = \frac{L^4}{12} = 1.33\;m^4$$

$$y_{p2} = 2.55 + \frac{1.33}{2.55x4} = 2.68\ m$$

Y1axe:

$$h_{p2} = y_{p2} sin75° = 2.59\ m$$

Effort F :

$$M(F_1)/A = F_1(y_{p1} - OA) = 11867.03\left(3.178 - \frac{2}{sin75°}\right) = 131384.3\ N.m$$

$$M(F_2)/A = F_2\left(y_{p2} - \frac{1.5}{sin75°}\right) = 94496.54\ N.m$$

Since:

$$M(F_1)/A > M(F_2)/A$$

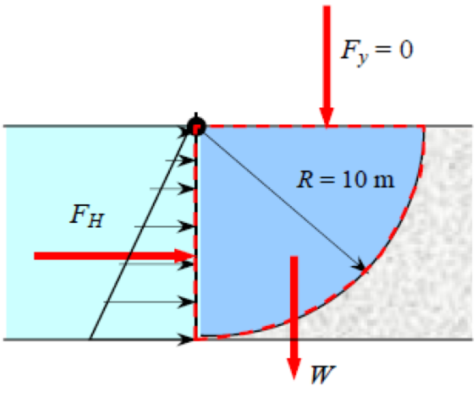
then the direction of force F is in the direction of force F2:

$$\sum M(F)/A = 0 \Longrightarrow FxAB + F_2\left(y_{p2} - \frac{1.5}{sin75°}\right) = F_1\left(y_{p1} - \frac{2}{sin75°}\right)$$

$$F = \frac{F_1\left(y_{p1} - \frac{2}{sin75°}\right) - F_2\left(y_{p2} - \frac{1.5}{sin75°}\right)}{2} = 18443.88\ N$$

Exercise N°17:

dam with a length of 100 m has a water-side surface shaped as a quarter circle of radius 10 m. The dam is filled to the rim. Determine the magnitude and line of action of the total hydrostatic force acting on this curved surface. Take the water density as 1000 kg/m³.



Solution N°17:

We analyze the free-body diagram of the water block bounded by the curved surface of the dam and its vertical and horizontal projections. The hydrostatic forces on each projected plane surface, along with the weight of the water block, are determined as follows:

Horizontal force on vertical surface:

$$F_H = F_x = P_{ave}A = \rho g h_c A = \rho g \left(\frac{R}{2}\right) A$$

$$F_H = \rho g \left(\frac{R}{2}\right) A = 1000x9.81x \left(\frac{10}{2}\right)(10x100) = 49050000\ N$$

The vertical force on the horizontal projection is zero, as it coincides with the free surface of the water. The weight of the water block per unit length is:

$$F_V = W = \rho g \forall = \rho g \left(Wx \frac{\pi R^2}{4}\right) = 1000x9.81x \left[100x\pi x \frac{10^2}{4}\right]$$

$$F_V = 77047560\ N$$

The magnitude and direction of the resultant hydrostatic force acting on the curved surface of the dam are then given by:

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{49050000^2 + 77047560^2} = 91335804\ N$$

$$tan\theta = \frac{F_V}{F_H} = \frac{77047560}{49050000} \longrightarrow \theta = 57.5°$$

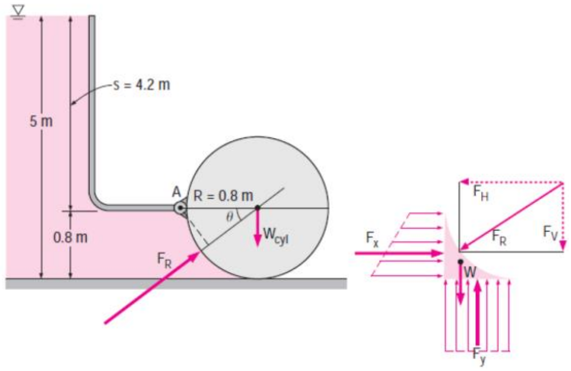
The resultant hydrostatic force therefore acts through the centre of curvature of the dam surface, directed at an angle of 57.5° below the horizontal.

Exercice N° 18 :

A solid cylinder of radius 0.8 m and of unit length is used as an automatic gate, hinged at point A. As shown in the free-body diagram, the gate is designed to open by rotating about the hinge when the upstream water level reaches 5 m. Determine:

(a) The resultant hydrostatic force acting on the cylinder and its line of action at the moment the gate opens.

(b) The weight of the cylinder per metre of length.



Solution N° 18:

(a) We analyze the free-body diagram of the water block bounded by the curved surface of the cylinder and its vertical and horizontal projections. The hydrostatic forces on each projected plane surface, together with the weight of the enclosed water block, are computed as follows:

Horizontal component — force on the vertical projection:

$$F_H = F_x = P_{ave}A = 9810x\left(4.2 + \frac{0.8}{2}\right)(0.8x1) = 36101\ N$$

Vertical force on horizontal surface (upward):

$$F_y = P_{ave}A = 9810x5(0.8x1) = 39240\ N$$

Weight of the water block per metre of length (acting downward):

$$W = mg = \rho g \forall = \rho g\left(R^2 - \frac{\pi R^2}{4}\right)$$

$$W = 9810\left(0.8^2 - 0.8^2\frac{\pi}{4}\right)(1) = 1347\ N$$

Therefore, the net upward vertical force is:

$$F_V = F_y - W = 39240 - 1347 = 37893\ N$$

The magnitude and direction of the resultant hydrostatic force acting on the curved surface of the cylinder are then:

$$F_R = \sqrt{F_H^2 + F_V^2}$$

$$F_R = \sqrt{36101^2 + 37893^2} = 52337\ N$$

the angle it makes with the horizontal is:

$$\theta = arctan\left(\frac{F_V}{F_H}\right) = arctan\left(\frac{37893}{36101}\right) \longrightarrow \theta = 46.4°$$

(b) At the moment the gate begins to open, the water level has reached 5 m and the contact force at the base of the cylinder vanishes. The only forces acting on the cylinder apart from the hinge reaction are its own weight, applied through its centre, and the resultant hydrostatic force exerted by the water. Setting the sum of moments about the hinge point A equal to zero yields:

$$F_R R sin\theta - W_{cyl} R = 0 \Longrightarrow W_{cyl} = F_R sin\theta = 52337 x sin 46.4 = 37900\ N$$

Chapter 3 :Ideal Incompressible Fluid Dynamics

Chapter Objectives

At the end of this chapter, the student should be able to:

- ✓ Understand The Fundamental Principles Of Mass Conservation Applied To Fluid Flow;
- ✓ Establish And Apply The Continuity Equation;
- ✓ Determine Volumetric And Mass Flow Rates In Different Flow Configurations;
- ✓ Apply Bernoulli's Equation To The Analysis Of Ideal And Real Fluid Flows;
- ✓ Analyze Energy Transfer In Hydraulic Systems Involving Head Losses And Hydraulic Machines;
- ✓ State And Apply Euler's Theorem;
- ✓ Evaluate The Interaction Forces Between A Moving Fluid And A Solid Surface Or An Obstacle.

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1 Introduction:

Hydrodynamics studies phenomena related to fluid flow. The dynamic behavior of fluids is governed by equations of motion. In this chapter, we will focus on perfect fluids where friction is neglected (zero viscosity) and density "ρ" remains constant (incompressible fluids).

Three types of forces are considered:

- ✓ Body forces, such as gravity
- ✓ Surface forces, such as pressure

Inertial forces, resulting from particle acceleration

This chapter emphasizes two fundamental relationships essential to fluid mechanics:

- ✓ The continuity equation (conservation of mass)
- ✓ Bernoulli's equation (conservation of energy)

A commonly studied case is when fluid properties such as density (ρ), pressure (P), and velocity (v) do not vary with time. This is known as steady or stationary flow (dv/dt = 0).

At the end of this chapter, the student should be able to:

- ✓ Write the continuity equation,
- ✓ Calculate mass flow rate and volumetric flow rate,
- ✓ Apply Bernoulli's theorem,
- ✓ State Euler's theorem and evaluate the interaction forces between a fluid and an obstacle.

2 Steady Flow, Streamlines, and Stream Tubes

Steady Flow : A fluid flow is considered steady if the velocity of the fluid particles remains constant over time at any given point. Note, however, that this does not imply the velocity field is uniform throughout space; velocity can still vary from one position to another.

Streamline : A streamline is a curve that is tangent at every point to the velocity vector of the fluid at that point. In steady flow, streamlines are stationary (invariant in the reference frame) and coincide with the actual pathlines (trajectories) of the fluid particles.

Stream Tube : A stream tube is defined as the surface formed by the set of streamlines passing through a closed contour (a closed loop).

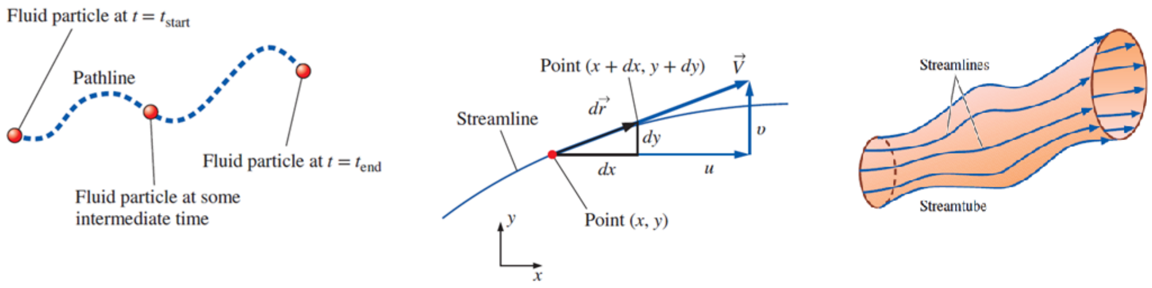


Figure 3. 1Pathlines, Streamlines, and Stream Tubes in Fluid Dynamics

3 Equation of continuity/ Conservation of Mass:

Consider a perfect incompressible fluid flowing through a pipe under steady-flow conditions, as illustrated in the figure below. In steady flow, the fluid properties at a given point do not vary with time.

To establish the continuity equation, the following notations are introduced:

- S1 and S2: inlet and outlet cross-sectional areas of the fluid at time t,
- S1’ and S2’ : positions of the inlet and outlet sections at the instant t’=t+dt
- $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$: average flow velocity vectors through sections S1 and S2 , respectively,
- $dx_1$and dx_2 :distances traveled by the fluid particles across sections S1 and S2 during the time interval dt,
- dm_1: elementary mass entering the control volume between S1 et S1’
- dm_2: elementary mass entering the control volume between S2 et S2’
- M: total mass of fluid contained between sections S1 and S2,
- dV_1: elementary volume entering the control volume between S1 et S1’
- dV_2: elementary volume entering the control volume between S2 et S2’

These quantities will be used to derive the mathematical expression of mass conservation for fluid flow.

At time t: The mass of fluid between sections S1 and S2 is $dm_1 + m$

At time t+dt: The mass of fluid between sections S1’ and S2’ is $dm_2 + m$

By the Law of Conservation of Mass, the mass within the system remains constant:

$$dm_1 + m = dm_2 + m \implies dm_1 = dm_2$$

Expressing mass in terms of density (ρ), cross-sectional area (S), and displacement (dx):

$$\rho_1 dV_1 = \rho_2 dV_2$$

$$\rho_1 . S1 . dx_1 = \rho_2 . S2 . dx_2$$

Dividing by the time interval dt, we obtain:

$$\rho_1 . S1 . \frac{dx_1}{dt} = \rho_2 . S2 . \frac{dx_2}{dt} \Leftrightarrow \rho_1 . S1 . v_1 = \rho_2 . S2 . v_2$$

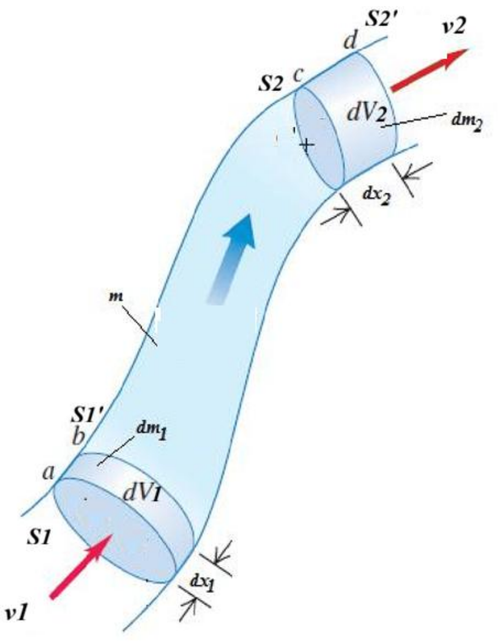
For an incompressible fluid, the density is constant :

$\rho_1 = \rho_2$.

The continuity equation simplifies to:

$$S1 . v_1 = \rho_2 . S2$$

This relationship defines the volume flow rate Q (expressed in m³/s). In steady, incompressible flow, the volume flow rate remains constant throughout the stream tube.



4 Mass Flow Rate and Volumetric Flow Rate

In fluid mechanics, flow rate is one of the fundamental quantities used to describe the movement of a fluid through a conduit or pipe. There are two distinct ways to quantify this movement, depending on whether we consider the *mass* or the *volume* of fluid crossing a reference surface per unit time.

4.1 Mass Flow Rate

The mass flow rate q_m is defined as the elementary mass *dm* of fluid crossing a straight elementary cross-section *dS* during an elementary time interval *dt*. It represents the quantity of matter (in kilograms) passing through a section per second.

$$q_m = \frac{dm}{dt}$$

Taking into account the continuity equation and the kinematic relations between velocity and displacement, one can show that the elementary mass of fluid crossing a section *S* during *dt* is:

$$q_m = \frac{dm}{dt} = \rho.S1.\frac{dx_1}{dt} = \rho.S2.\frac{dx_2}{dt}$$

$$q_m = \rho.S1.v_1 = \rho.S2.v_2$$

For any straight cross-section *S* of the conduit, where the fluid flows at mean velocity *v*, the general expression of the mass flow rate is:

$$q_m = \rho.S.v$$

Note:

The equality $q_m = \rho \cdot v_1 \cdot S_1 = \rho \cdot v_2 \cdot S_2$ is a direct consequence of mass conservation (continuity equation): in steady, incompressible flow, the mass flow rate is constant along the conduit. If the cross-section decreases, the velocity increases proportionally.

Important Remark: For compressible fluids (gases at high velocities), ρ varies along the conduit and the relation must be applied locally.

4.2 Volumetric Flow Rate (Q_v)

The volumetric flow rate Q_v (also called *volume flow rate* or *discharge*) is the elementary volume *dV* of fluid crossing a straight elementary surface *dS* during a time interval *dt*. It expresses what volume of fluid passes through a section per second.

$$q_V = \frac{dV}{dt}$$

The elementary volume swept by the fluid through section *S* during *dt* is :

$dV = S.dx$,

where $dx = v.dt$ is the displacement of the fluid element.

Dividing by dt:

$q_V = \frac{dV}{dt} = S\frac{dx}{dt} = S.v$ [m/s³]

4.3 Relationship Between Mass Flow Rate and Volumetric Flow Rate

The density ρ of a fluid is defined as the ratio of mass to volume:

$$\rho = \frac{dm}{dV}$$

This fundamental definition directly links the two flow rates. Since :

$q_m = \frac{dm}{dt}$ and $q_V = \frac{dV}{dt}$

dividing gives:

$$q_m = \rho.q_V$$

Key Relationship

For an incompressible fluid (constant density ρ), the volumetric flow rate Q_v is also conserved along the conduit ($Q_v = v_1S_1 = v_2S_2$). This is the basis of many hydraulic calculations.

For compressible fluids (gases), only the mass flow rate q_m is conserved, since ρ may vary with pressure and temperature. The volumetric flow rate must always be specified at a reference condition (temperature and pressure).

5 Bernoulli's theorem

Bernoulli's theorem expresses the conservation of fluid energy. Let's consider a perfect, incompressible fluid flowing in a permanent stream, subject only to the forces of gravity as shown in Figure 3. 2 below.

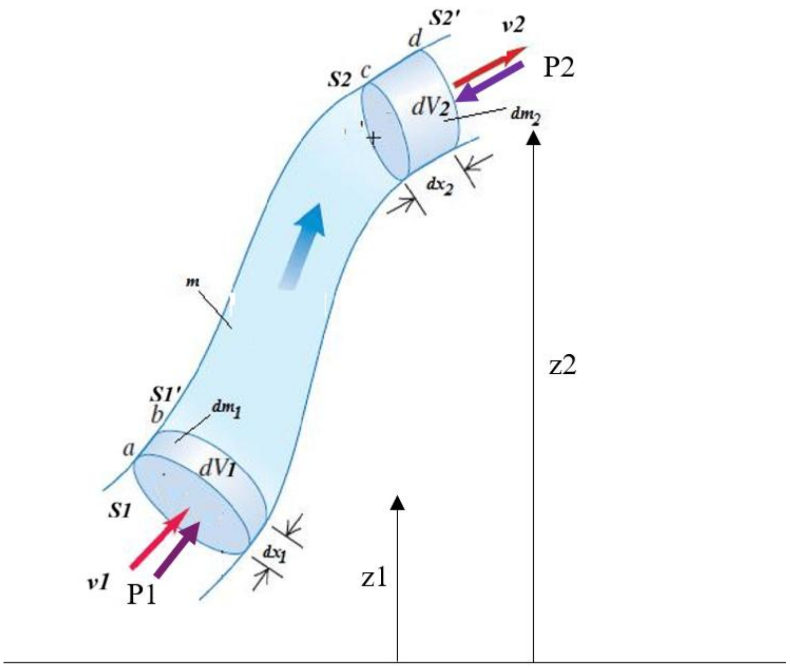


Figure 3. 2 The forces acting on a fluid particle along a streamline.

Consider a steady flow of an incompressible and ideal fluid through a pipe (Figure 3. 2). Between two sections S1 and S2, an elementary fluid mass dm moves along the streamline. The elevations of the centers of gravity are denoted by z1 and z2, while the flow velocities are v1 and v2. The pressures acting on the two sections are P1 and P2, respectively.

Applying the kinetic energy theorem to the fluid element, the variation of kinetic energy is equal to the sum of the external forces work:

$$\Delta E_C = \sum W$$

The change in kinetic energy between the two sections is expressed as:

$$\Delta E_C = \frac{1}{2} dm(v_2^2 - v_1^2)$$

The forces acting on the fluid are:

Gravity force: $W_P = (Z1 - Z2).g.dm$

Pressure forces: On surface $S1$: $W_{P1} = P1.S1.v_1 dt$

On surface $S2$: $W_{P2} = -P2.S2.v_2 dt$

On the lateral surface: $W_{Pl} = 0$

For an ideal fluid, internal friction forces are neglected $\mu = 0$.

$\Delta E_C = \sum W \Longrightarrow \frac{1}{2} dm(v_2^2 - v_1^2) = P1.S1.v_1 dt - P2.S2.v_2 dt + (Z1 - Z2).g.dm$

$\Longrightarrow \frac{1}{2} dm.\rho.(v_2^2 - v_1^2) = P1.\rho.S1.v_1 dt - P2.\rho.S2.v_2 dt + (Z1 - Z2).\rho.g.dm$

Using the continuity equation:

$$dm = \rho.S1.v_1.dt = \rho.S2.v_2.dt$$

$$\frac{1}{2} dm.\rho.(v_2^2 - v_1^2) = P1.dm - P2.dm + (Z1 - Z2).\rho.g.dm$$

Bernoulli’s equation is obtained as follows:

$$P1 + \rho.g.z1 + \frac{1}{2}\rho v_1^2 = P2 + \rho.g.z2 + \frac{1}{2}\rho v_2^2$$

This equation shows that, for an ideal fluid flow, the sum:

$$P + \rho.g.z + \frac{1}{2}\rho v^2 = Cst$$

remains constant along a streamline.

The terms of Bernoulli’s equation represent:

- ✓ P: static pressure of the fluid;
- ✓ $\rho.g.z$: hydrostatic pressure: gravitational potential energy per unit volume;
- ✓ $\frac{1}{2}\rho v^2$: dynamic pressure : kinetic energy per unit volume, also called dynamic pressure.

Total Pressure: The sum of static, dynamic, and hydrostatic pressures. Bernoulli's principle states that total pressure remains constant along a streamline.

Therefore, Bernoulli’s theorem expresses the conservation of mechanical energy per unit volume in fluid flow.

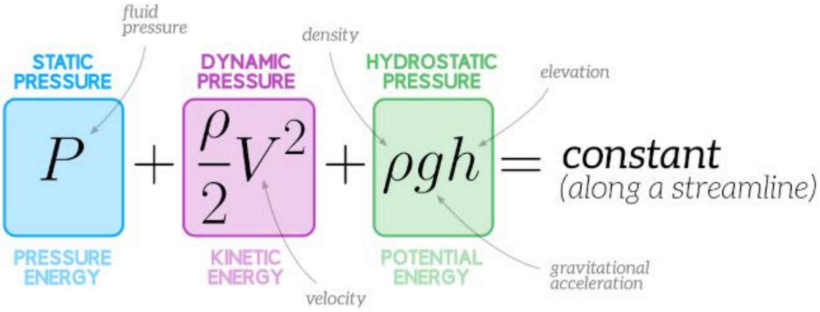


Figure 3. 3 Bernoulli’s Equation Interpretation

5.1 Alternative Forms of Bernoulli's Equation

Bernoulli's equation, originally expressed in terms of pressure (Pa or N/m²), can be rewritten in several ways depending on the application.

5.1.1 Energy per Unit Weight (Head Form)

By dividing the standard expression :

$$P + \rho . g . z + \frac{1}{2}\rho v^2 = Cst$$

by the specific weight $\rho . g$., we obtain:

$z + \frac{P}{\rho . g} + \frac{v^2}{2g} = Cst$ [m]

In this form, each term represents a head (height) measured in meters [m].

This is also equivalent to energy per unit weight.

- ✓ z : Elevation head.
- ✓ $\frac{v^2}{2g}$: Velocity head.
- ✓ $\frac{P}{\rho . g}$: Pressure head

5.1.2 Energy per Unit Mass

By dividing the standard expression by the density ρ, we obtain:

$$\frac{P}{\rho} + g . z + \frac{1}{2} v^2 = Cst$$

In this case, all terms represent energy per unit mass, expressed in Joules per kilogram [J/kg].

6 Generalized Bernoulli Equation with a Hydraulic Machine

Let us consider again the fluid stream between sections S1 and S2, under the same assumptions of steady incompressible ideal fluid flow. In addition, a hydraulic machine is installed between these two sections.

This machine can be either:

- a **pump**, which supplies energy to the fluid;
- a **turbine**, which extracts energy from the fluid.

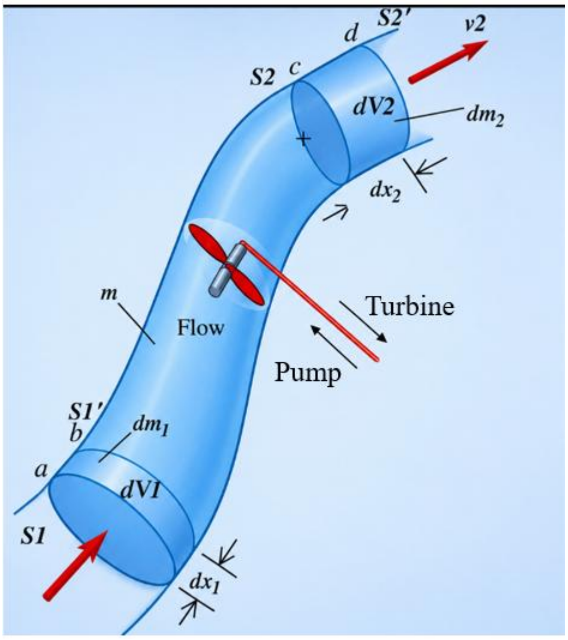


Figure 3. 4 Schematic illustration of Bernoulli's equation applied to a fluid flow with energy exchange through a pump or a turbine.

The machine is characterized by:

- the power exchanged with the fluid P;
- the shaft power Pa;
- the efficiency η.

As the fluid passes through the hydraulic machine, energy is exchanged in the form of work ΔW during a time interval Δt. The exchanged power is defined by:

$P = \frac{\Delta W}{\Delta t}$ [J/s]=[Watt]

- ✓ If P>0 , the machine supplies energy to the fluid: the machine acts as a pump;
- ✓ If P<0, the machine extracts energy from the fluid: the machine acts as a turbine.

6.1.1 Pump efficiency

For a pump, the efficiency is defined as the ratio between the hydraulic power transmitted to the fluid and the mechanical shaft power supplied to the machine:

$$\eta = \frac{P}{Pa}$$

6.1.2 Turbine efficiency

For a turbine, the efficiency is the ratio between the mechanical power recovered on the shaft and the hydraulic power supplied by the fluid:

$$\eta = \frac{Pa}{P}$$

Applying Bernoulli's equation between sections S1 and S2, while considering the energy exchange with the hydraulic machine, gives:

$$P1 + \rho.g.z1 + \frac{1}{2}\rho v_1^2 = P2 + \rho.g.z2 + \frac{1}{2}\rho v_2^2 \pm \frac{P}{q_V}$$

$\frac{P}{q_V}$: quantity, expressed in Pascals (Pa), corresponds to the energy per unit volume supplied to the fluid by a pump or extracted from the fluid by a turbine.

7 Applications of Bernoulli's Theorem

The Bernoulli equation is used to analyze various devices and perform measurements in fluid mechanics. Below are some key applications.

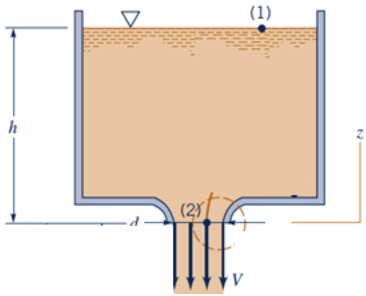
7.1 Water Discharge from a Large Tank/ Draining Velocity of a Tank/ Torricelli's Theorem

One of the most straightforward applications of Bernoulli's equation is the determination of the outflow velocity through a small orifice at the base (or side) of an open-surface reservoir.

Consider a large tank with a free surface (point 1) and a small circular orifice at the bottom (point 2). The following assumptions apply:

- The fluid is ideal (inviscid) and incompressible
- The flow is steady
- The orifice cross-section S_2 is much smaller than the tank cross-section S_1 : $S_2 \ll S_1$
- As a consequence, the free-surface velocity is negligible: $v_1 \approx 0$
- Both the free surface and the orifice are exposed to atmospheric pressure: $P_1 = P_2 = Patm$

Applying Bernoulli's equation between points 1 and 2:



$$P1 + \rho.g.z1 + \frac{1}{2}\rho v_1^2 = P2 + \rho.g.z2 + \frac{1}{2}\rho v_2^2$$

Cancelling the pressure terms and setting $v_1 \approx 0$, with $z1 - z2 = h,$ the height of fluid above the orifice:

$$\rho.g.h = \frac{1}{2}\rho v_2^2$$

$$v_2 = \sqrt{2.g.h}$$

This result is known as Torricelli's theorem: the outflow velocity equals the free-fall velocity of a body dropped from the height h of the free surface above the orifice. It depends only on the fluid height and is independent of the fluid density.

7.2 Venturi Meter/ Flow Rate Measurement

The Venturi tube is a device designed to measure the volumetric flow rate of a fluid by exploiting the pressure drop that occurs when the flow is forced through a converging section (throat).

When a fluid passes through a constriction, its velocity must increase (continuity equation: $S_1v_1 = S_2v_2$). By Bernoulli's equation, this velocity increase is accompanied by a local pressure drop. Measuring this pressure difference between the wide inlet section and the narrow throat is therefore sufficient to determine the flow rate.

The device consists of three sections:

- A converging inlet that smoothly accelerates the flow from section S_1 to the throat S_2
- A throat (minimum cross-section) where velocity is maximum and pressure is minimum
- A diverging outlet (diffuser) that gradually decelerates the flow and recovers most of the pressure

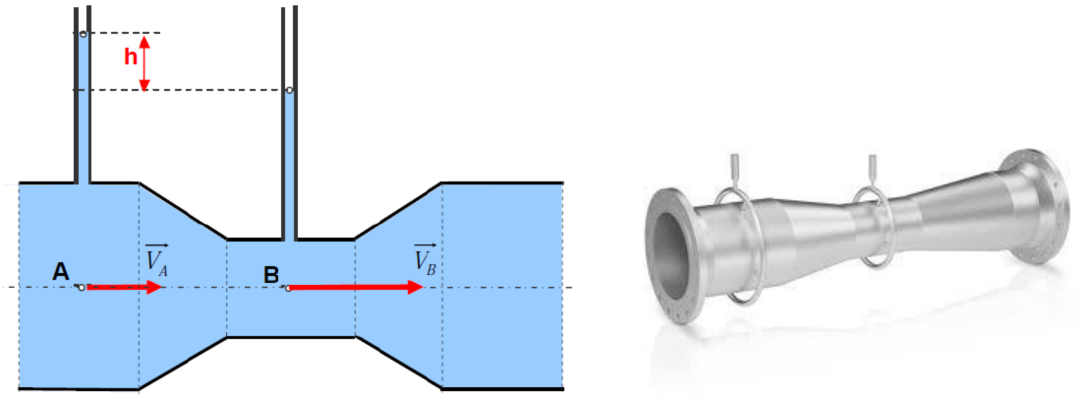


Figure 3. 5 venturi meter

Applying Bernoulli's equation between points A (inlet) and B (throat):

$$PA + \rho.g.zA + \frac{1}{2}\rho v_A^2 = PB + \rho.g.zB + \frac{1}{2}\rho v_B^2$$

Considering the following conditions:

- **Horizontal alignment:** zA=zB (same elevation level).
- **Continuity equation:** $v_A.SA = v_B.SB$
- **Hydrostatic pressure difference:** $PA - PB = \rho.g.h$

By combining these equations, we derive the expression for the fluid velocity at the throat (constriction):

$$v_B = \sqrt{\frac{2gh}{1 - \left(\frac{SB}{SA}\right)^2}}$$

The theoretical flow rate is then given by:

$$Q = SB.v_B$$

In practice, this value is considered a theoretical flow rate because it assumes an ideal (inviscid) fluid. To obtain the actual flow rate, we multiply the theoretical value by a correction factor that accounts for energy losses within the Venturi meter. This factor is known as the discharge coefficient (Cd):

$$Q = Cd.SB.v_B$$

7.3 The Pitot Tube

Henri Pitot (French engineer and physicist, 1695-1771) designed a device (Figure 3. 6) that is still used to determine aircraft speed, but also to measure flow velocity in the laboratory. Like the Venturi tube, it uses the pressure variations described by Bernoulli's law. The Pitot tube, often simply called a "Pitot," is the most widely used instrument for measuring fluid velocity in various flow conditions. Its operating principle is based on measuring the static pressure and the stagnation pressure at a specific point in the flow. The device is named in honor of its inventor, Henri de Pitot, who first tested the instrument in the Seine River in August 1732.

Consider a fluid in steady flow within a pipe. Two tubes are immersed in the liquid:

- Tube A (Stagnation point): The opening faces the current. At this point, the fluid is brought to a complete stop, meaning the velocity is zero ($v_A = 0$). The pressure measured here is the stagnation pressure PA.
- Tube B (Static point): The opening is parallel to the streamlines. The fluid velocity v at this point is the same as in the pipe, and the pressure measured is the static pressure PB.

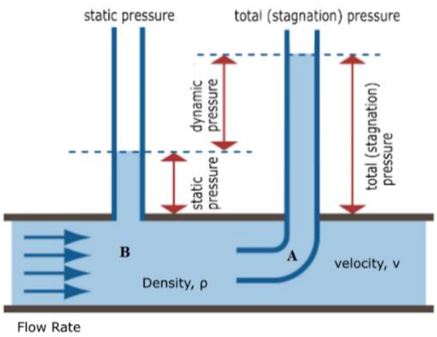


Figure 3. 6 Pitot Tube

Applying Bernoulli's equation between points **A** (stagnation) and **B** (static):

$$PA + \rho.g.zA + \frac{1}{2}\rho v_A^2 = PB + \rho.g.zB + \frac{1}{2}\rho v_B^2$$

Given the following conditions:

- Same elevation: zA=zB
- Stagnation point: $v_A = 0$

The equation simplifies to:

$$\frac{PA - PB}{\rho.g} = \frac{v_B^2}{2g} \quad \Rightarrow \; v_B = \sqrt{\frac{2(PA - PB)}{\rho}}$$

From hydrostatic principles, the pressure difference is related to the height h in the columns:

$$PA - PB = \rho.g.h$$

Substituting this into the velocity equation, we obtain the final expression for the fluid velocity:

$$v_B = \sqrt{2gh}$$

7.4 Diaphragm

The diaphragm is a device widely used in fluid mechanics for flow rate measurement in hydraulic pipelines. It generally consists of a thin plate perforated with a circular orifice and positioned perpendicular to the fluid flow.

When a fluid passes through the diaphragm, the flow cross-sectional area decreases abruptly. This reduction causes:

- an increase in the fluid velocity;
- a decrease in the static pressure;
- a head loss in the pipeline.

The pressure difference created between the upstream and downstream sections of the diaphragm makes it possible to experimentally determine the flow rate of the fluid circulating in the pipe.

Unlike the Venturi tube, the diaphragm produces a sudden contraction of the flow section (Figure 3. 7), resulting in significant head losses. In contrast, the Venturi tube has a gradual converging–diverging shape that reduces energy dissipation and therefore produces lower head losses.

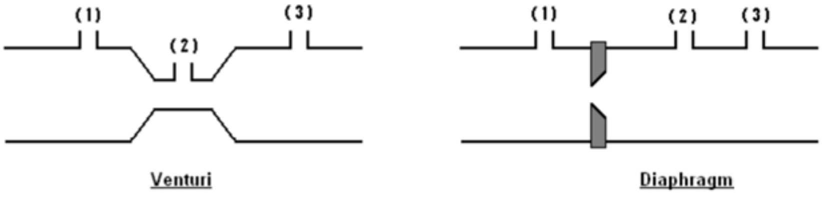


Figure 3. 7 Flow Measurement: Venturi Tube vs. Diaphragm

The study aims to establish a relationship between the water flow rate through the orifice and the differential water level h in the tubes.

For an ideal incompressible fluid, the volumetric flow rate is constant across all cross-sections of a pipe:

$$Q_V = v_1 S1 = v_2 S2 = vS$$

From this, the upstream velocity can be expressed in terms of the throat velocity:

$$v_1 = v\frac{S}{S1}$$

Applying the Bernoulli's equation between points A (upstream) and C (at the orifice throat):

$$\frac{v_1^2}{2g} + \frac{P_1}{\rho g} + z_1 = \frac{v^2}{2g} + \frac{P}{\rho g} + z$$

Since the pipe is horizontal, $z_1 = z_2 = z$, which simplifies to:

$$\frac{v_1^2}{2g} + \frac{P_1}{\rho g} = \frac{v^2}{2g} + \frac{P}{\rho g}$$

Rearranging gives the pressure head difference:

$$\frac{P_1 - P}{\rho g} = \frac{v^2 - v_1^2}{2g} = h$$

where h is the difference in water column height observed in the manometer tubes upstream and downstream of the orifice.

Combining Bernoulli's equation and the continuity equation yields the theoretical volumetric flow rate through the orifice:

$$Q = S.v$$

Starting from:

$$v^2 - v_1^2 = 2.g.h$$

Substituting $v_1 = v\frac{S}{S1}$ into the above:

$$v^2 - \left(\frac{S}{S1}v\right)^2 = 2.g.h$$

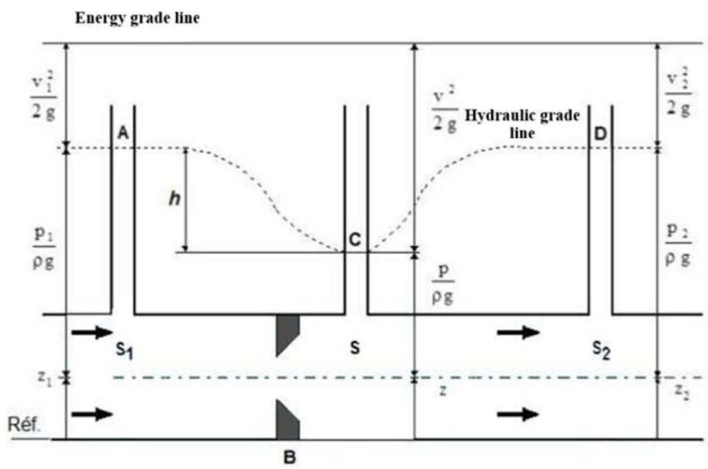
$$v^2\left(1-\left(\frac{S}{S1}v\right)^2\right)=2.g.h$$

Solving for the throat velocity:

$$v=\sqrt{\frac{2.g.h}{1-\left(\frac{S}{S1}\right)^2}}$$

The theoretical flow rate is therefore:

$$Q=\frac{S}{\sqrt{1-\left(\frac{S}{S1}\right)^2}}\sqrt{2.g.h}$$



8 Geometrical Interpretation of Bernoulli's Equation

The graphical representation of these energy terms leads to the definition of two important lines:

Hydraulic Grade Line (HGL)

The Hydraulic Grade Line (HGL) represents the sum of the pressure head and the elevation head:

HGL= $\frac{P}{\rho g}+z$

It corresponds to the level reached by the fluid in piezometric tubes connected to the flowing fluid.

- ✓ Energy Grade Line (EGL)

The Energy Grade Line (EGL) represents the total mechanical energy of the fluid:

EGL=$\frac{V^2}{2g} + \frac{P}{\rho g} + z$

The vertical distance between the EGL and the HGL is equal to the velocity head:

EGL-HGL=$\frac{V^2}{2g}$

For an ideal Bernoulli flow without friction losses, the total head remains constant; therefore, the EGL is horizontal along the flow direction. The HGL may rise or fall depending on the variation of the flow velocity inside the fluid stream.

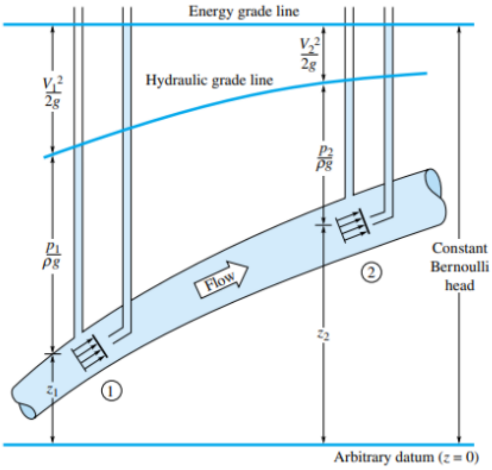


Figure 3. 8 Representation of Bernoulli Equation

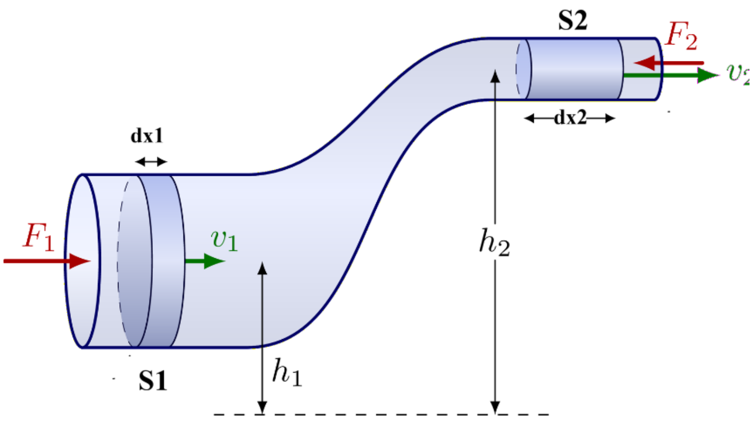
9 Euler's Momentum Theorem

While Bernoulli's equation is a powerful tool for relating pressure and velocity along a streamline, its scope is fundamentally limited — it cannot account for the mechanical forces exchanged between a fluid and a solid boundary. To evaluate such forces, a second, more general theorem is required.

A direct and important application of Euler's theorem is the calculation of forces exerted by fluid jets on solid surfaces. These forces are exploited in a wide range of engineering systems, including hydroelectric turbines that convert hydraulic energy into electricity, water-jet cutting machines, and many other industrial devices.

Consider a portion of an incompressible fluid flowing steadily through a pipe. This fluid element is bounded by a closed control surface consisting of:

- an inlet cross-section S1, through which fluid enters with velocity v_1,
- an outlet cross-section S2, through which fluid exits with velocity v_2.



Euler's theorem is derived from Newton's second law applied to the fluid system — the fundamental principle of dynamics:

$$\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}$$

where $\vec{P}$ is the total linear momentum of the fluid system. The change in momentum over an infinitesimal time interval dtdt dt is:

$$d\vec{P} = d(m.d\vec{v}) = m_2.\overrightarrow{v_2} - m_1.\overrightarrow{v_1}$$

Since the infinitesimal fluid volumes at inlet and outlet are $V_1 = S_1.dx_1$ and $V_2 = S_2.dx_2$, and the fluid has uniform density ρ, the masses can be written as $m_i = \rho.qV_i$. Substituting:

$d\vec{P} = \rho.S_2.dx_2\overrightarrow{v_2} - \rho.S_1.dx_1\overrightarrow{v_1}$

Noting that dx_i, this becomes:

$d\vec{P} = \rho.S_2.v_2dt\overrightarrow{v_2} - \rho.S_1.v_1dt\overrightarrow{v_1}$

From the continuity equation for incompressible flow, the volumetric flow rate is constant:

$$Q_V = S_2.v_2 = S_1.v_1$$

Substituting this simplification:

$$d\vec{P} = \rho.Q_V.dt.\overrightarrow{v_2} - \rho.Q_V.dt.\overrightarrow{v_1}$$

Dividing both sides by dtdt dt:

$$\frac{d\vec{P}}{dt} = \rho.Q_V.(\overrightarrow{v_2} - \overrightarrow{v_1})$$

Since the mass flow rate is $q_m = \rho Q_V$, Euler's theorem takes its final form:

$$\sum \overrightarrow{F_{ext}} = q_m(\overrightarrow{v_2} - \overrightarrow{v_1})$$

The term $\sum \overrightarrow{F_{ext}}$ represents the vector sum of all external forces acting on the isolated fluid element between S_1 and S_2 . These forces include:

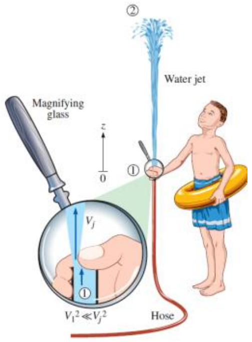
- ✓ Weight of the fluid contained between S_1, S_2 , and the pipe wall;
- ✓ Pressure forces acting on the inlet section S_1 , the outlet section S_2 , and the pipe walls;
- ✓ Reaction forces exerted by the pipe structure on the fluid.

PRACTICAL EXERCISES

Exercise N°1

A garden hose is connected to a municipal water supply line maintained at a gauge pressure of 400 kPa. While the hose is pointed vertically upward, a child partially blocks the outlet with their thumb, causing the water to exit as a narrow, high-velocity jet.

Assuming ideal flow conditions, determine the maximum height the water jet can reach.



Solution N°1 :

The water height will be maximum under the stated assumptions. The velocity inside the hose is relatively low (V1= 0) and we take the hose outlet as the reference level (z1=0). At the top of the water trajectory V2= 0, and atmospheric pressure pertains. Then the Bernoulli equation simplifies to:

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

The flow velocity within the hose is assumed negligible in comparison to the exit velocity of the jet, and is therefore taken as zero : $V_1 : ignore$

The reference elevation is set at the hose outlet, so that $z_1 = 0$

At the apex of the jet's trajectory, the fluid velocity vanishes $V_2 = 0$ and the pressure returns to atmospheric. Applying Bernoulli's equation along a streamline from point 1 to point 2 then reduces to:

$$\frac{P_1}{\rho g} = \frac{P_{atm}}{\rho g} + z_2$$

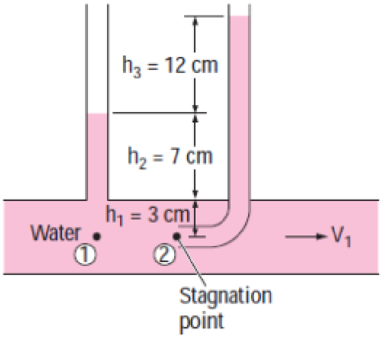
Solving for *z*2 and substituting,

$$z_2 = \frac{P_1 - P_{atm}}{\rho g} = \frac{P_{1,gage}}{\rho g} = \frac{400.10^3}{1000.9.81} = 40.8\,m$$

Exercise N°2

In a horizontal water pipe, both a piezometer and a Pitot tube are installed to measure pressures. The piezometer measures the static pressure, while the Pitot tube measures the stagnation pressure (sum of static and dynamic pressures).

Using the water column heights indicated in the piezometer and Pitot tube, determine the flow velocity at the center of the pipe. You may assume:The flow is steady,The fluid (water) is incompressible,The pipe is horizontal,Atmospheric pressure acts on both water columns.



Solution N°2:

We consider two points, 1 and 2, along the centerline of the pipe. Point 1 is directly beneath the piezometer, and point 2 is at the tip of the Pitot tube. Assuming steady flow with straight and parallel streamlines, the gage pressures at points 1 and 2 can be expressed as.

$P_1 = \rho g(h_1 + h_2)$; $P_2 = \rho g(h_1 + h_2 + h_3)$

Noting that point 2 is a stagnation point and thus $V_2 = 0$ and $z_1 = z_2$, the application of the Bernoulli equation between points 1 and 2 gives

$$\frac{P_1}{\rho g} + \frac{V_1^2}{\rho g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{\rho g} + z_2 \Rightarrow \frac{V_1^2}{\rho g} = \frac{P_2 - P_1}{\rho g}$$

Substituting the P_1 and P_2 expressions gives:

$$\frac{V_1^2}{\rho g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

Solving for V_1 and substituting,

$$V_1 = \sqrt{2x9.81x0.12} = \frac{1.53m}{s}$$

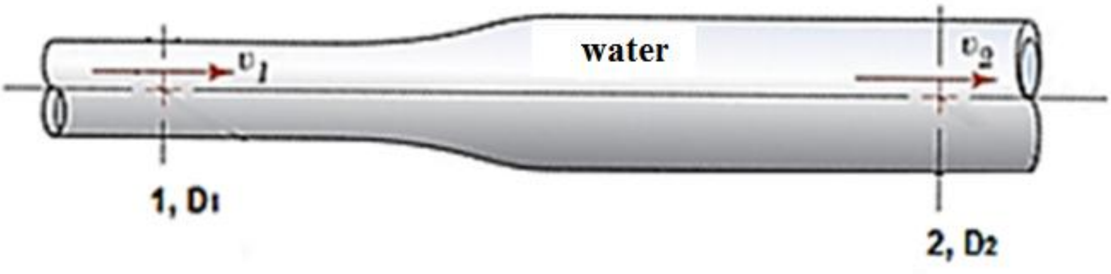
Note that to determine the flow velocity, all we need is to measure the height of the excess fluid column in the Pitot tube.

Exercise N°3:

A pipe has two different cross-sections (section 1 and section 2) with the following specifications:

- Diameter at section 1: 50 mm
- Diameter at section 2: 100 mm
- Water temperature: 70°C
- Average velocity at section 1: 8.0 m/s

Calculate: a) The water velocity at section 2 b) The volumetric flow rate through the pipe c) The corresponding mass flow rate



Solution N°3:

a. Speed in section 2

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1}{A_2} v_1$$

$$A_1 = \frac{\pi D_1^2}{4} = \frac{\pi 50^2}{4} = 1963mm^2$$

$$A_2 = \frac{\pi D_2^2}{4} = \frac{\pi 100^2}{4} = 7854mm^2$$

$$v_2 = v_1 \left(\frac{A_1}{A_2}\right) = 8x\frac{1963}{7854} = \frac{2m}{s}$$

b. Volume flow

Due to the principle of continuity, we can use the conditions of section 1 or section

section 2 to calculate QV. In section 1, we have :

$$Q_V = A_1 v_1 = 1963x10^{-3}x8 = 0.00157\frac{m^3}{s}$$

c. Mass flow rate

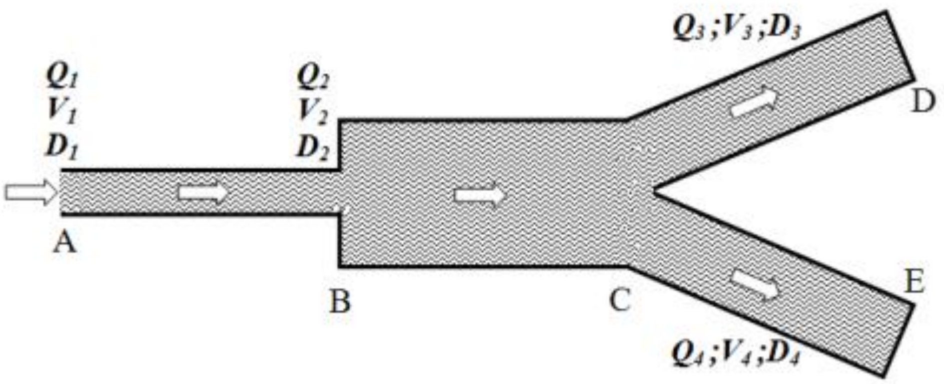
At 70°C, the density of water is 978 kg/m3 , so the mass flow rate is,

$$Q_m = \rho Q_V = 978x0.0157 = 15.36\frac{kg}{s}$$

Exercice N°4:

Water flows from point A to points D and E, as shown in the figure below. The system has four different sections (D1, D2, D3 and D4). Some flow parameters are given in the table below. Determine the missing parameters:

Section	Diameter (mm)	Flow m^3/s	Speed m/s
AB	300	?	?
BC	600	?	1,2
CD	?	$Q_3=2Q_4$	1,4
CE	150	$Q_4=Q_3/2$	?



Solution N°4:

$$Q_{V1} = V_1.A_1 = V_2.A_2 = Q_{V2}$$

$$Q_{V2} = V_2.A_2 = 1.2x\frac{\pi d_2^2}{4} = 1.2x\frac{3.14x0.6^2}{4} = 0.33912\frac{m^3}{s} = Q_{V1}$$

According to the figure and the data in the table, $A_2 = A_1x4$ or : $A_1 = \frac{A_2}{4}$

We replace by$A_1 = \frac{A_2}{4}$ in relation (1), we find :

$$V_1\frac{A_2}{4} = V_2xA_2 \Rightarrow \frac{V_1}{4} = V_2 \longrightarrow V_1 = 4xV_2 = 4x1.2 = 4.8\frac{m}{s}$$

$Q_{V2} = V_2.A_2 = Q_{V3} + Q_{V4}$ but: $Q_{V4} = \frac{Q_{V3}}{2}$

Therefore:

$$Q_{V2} = Q_{V3} + \frac{Q_{V3}}{2} = \frac{3}{2}Q_{V3}$$

$$V_2.A_2 = \frac{3}{2}V_3.A_3 \longrightarrow A_3 = \frac{2}{3}\frac{Q_{V2}}{V_3} = \frac{2x0.33912}{3x1.4} = 0.16148571\ m^2$$

From :

$$A_3 = \frac{\pi d_3^2}{4} \longrightarrow d_3 = \sqrt{\frac{4xA_3}{3.14}} = \sqrt{\frac{4x0.1615}{3.14}} = 0.45\ m$$

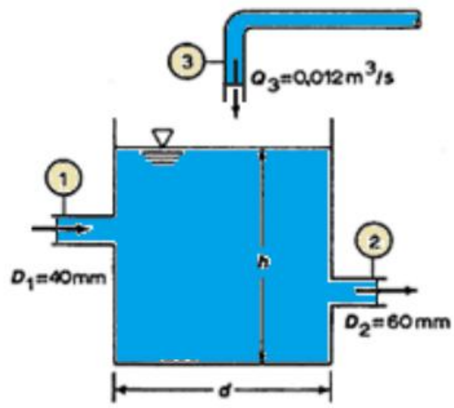
and likewise:

$$V_4 = \frac{V_3.A_3}{2A_4} = \frac{1.4x3.14xd_3^2}{2x3.14xd_3^2} = 6.3\frac{m}{s}$$

Section	Diameter (mm)	Flow ($\frac{m^3}{s}$)	Speed (m/s)
AB	300	0.33912	4.8
BC	600	0.33912	1.2
CD	450	$Q_3 = 2Q_4$	1.4
CE	150	$Q_4 = \frac{Q_3}{2}$	6.3

Exercise N°5:

The water tank in the figure below is filled by two inlets: section 1 with a velocity of 5 m/s and section 3 with a volumetric flow rate of 0.012 m^3/s. If the water level h remains constant, determine the outlet velocity V2.



Solution N°5:

By conservation of mass, we have :

$$Q_{V1} + Q_{V3} = Q_{V2}$$

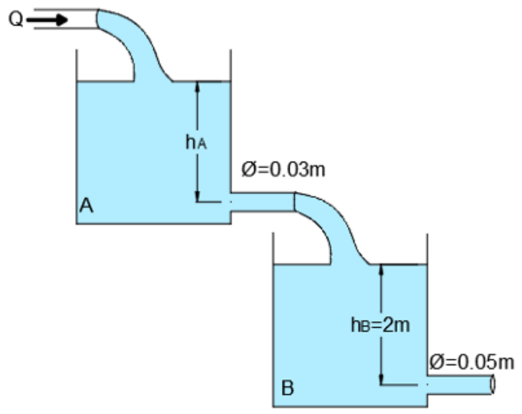
$$Q_{V2} = \frac{\pi x 0.04^2}{4} x5 + 0.012 = 0.01828 \frac{m^3}{s}$$

$$Q_{V2} = V_2 . A_2$$

$$V_2 = \frac{Q_{V2}}{A_2} = \frac{0.01828}{\frac{\pi x 0.06^2}{4}} = 6.47 \frac{m}{s}$$

Exercise N°6:

Water flows regularly into the large tanks A and B shown in the figure. Determine the height of the water hA. Given: the density of water is 1000 kg/m3 , gravity is 9.81 m/s^2.



Solution N°6:

According to the constant height hB.:

$$Q_2 = Q_4$$

And $Q_4 = V_4 . A_4$

With

$$\frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3 = \frac{P_4}{\rho g} + \frac{V_4^2}{2g} + z_4$$

Where , $P_3 = P_4 = 0$ and $V_3 = 0$

$$V_4 = \sqrt{2g(z_3 - z_4)} = \sqrt{2x9.81x2} = 6.26\frac{m}{s}$$

Where:

$$Q_4 = \frac{\pi}{4}(0.05)^2(6.26) = 0.0123\frac{m^3}{s}$$

Also:

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Where: $P_1 = P_2 = 0$ and $V_1 = 0$

$$V_1 = \sqrt{2gh_A}$$

So, $V_2 . A_2 = Q_4$

$$\frac{\pi}{4}(0.03)^2\sqrt{2x9.81xh_A} = 0.0123\frac{m^3}{s}$$

Hence,

$$h_A = 15.4\ m$$

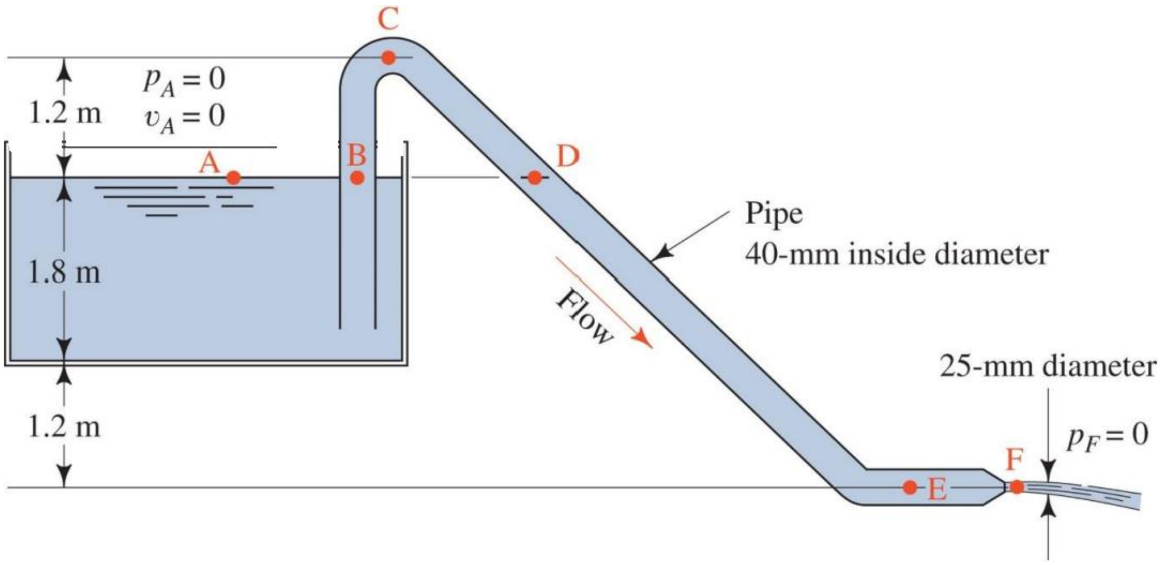
Exercice N°7:

In the siphon shown in the figure below (D=40 mm and d=25 mm), neglecting friction and considering the large reservoir, find: 1) the water flow in the tube at point F 2) the water velocity in the tube at D 3) the water pressure at D. We give :

$g = 9.81\ m/s^2$

$\rho_{H2O} = 1000\ kg/m^3$

$P_{atm} = 101325$ Pa.



Solution N°7:

1- Flow at point F

$$Q_{vF} = V_F . A_F$$

$$A_F = \frac{\pi d^2}{4} = \frac{3.14x(0.025)^2}{4} = 4.9x10^{-4}m^2$$

To obtain V_F, we apply Bernoulli's theorem between points A and F.

$$\frac{P_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{P_F}{\rho g} + \frac{V_F^2}{2g} + z_F$$

$$V_A = 0$$

And since the reservoir is open-air, we have :

$$P_A = P_F = P_{atm}$$

Then the previous Bernoulli equation can be simplified as follows:

$$z_A = \frac{V_F^2}{2g} + z_F$$

so:

$$V_F = \sqrt{2g(z_A - z_F)} = \sqrt{2x9.81(3)} = 7.67\frac{m}{s}$$

then

$$Q_{VF} = V_F.A_F$$

$$Q_{VF} = 7.67x4.9x10^{-4} = 3.75x10^{-3}\frac{m^3}{s}$$

$$Q_{VF} = 3.75\frac{l}{s}$$

2- Velocity at point D

By conservation of flow in the system, we write :

$$Q_{VB} = V_B.A_B = V_F.A_F$$

With, $A_B = \frac{\pi D^2}{4} = \frac{3.14x0.04^2}{4} = 1.26x10^{-3}\ m^2$

$$V_B = \frac{Q_{VB}}{A_B} = \frac{3.75x10^{-3}}{1.26x10^{-3}} = 3\frac{m}{s}$$

3- Water pressure in point D To get, we apply Bernoulli's theorem between points D and F.

$$\frac{P_D}{\rho g} + \frac{V_D^2}{2g} + z_D = \frac{P_F}{\rho g} + \frac{V_F^2}{2g} + z_F$$

$$P_D = \rho g\left(\frac{P_F}{\rho g} + \frac{V_F^2}{2g} + z_F - \frac{V_D^2}{2g} - z_D\right)$$

$$P_D = 1000x9.81\left(\frac{101325}{1000x9.81} + \frac{58.82}{2x9.81} + 0 - \frac{9}{2x9.81} - 3\right) = 96903.18\ Pa$$

Exercise N°8:

In a Venturi-type nozzle (see figure below), water flows steadily through the pipe. The following assumptions are considered:

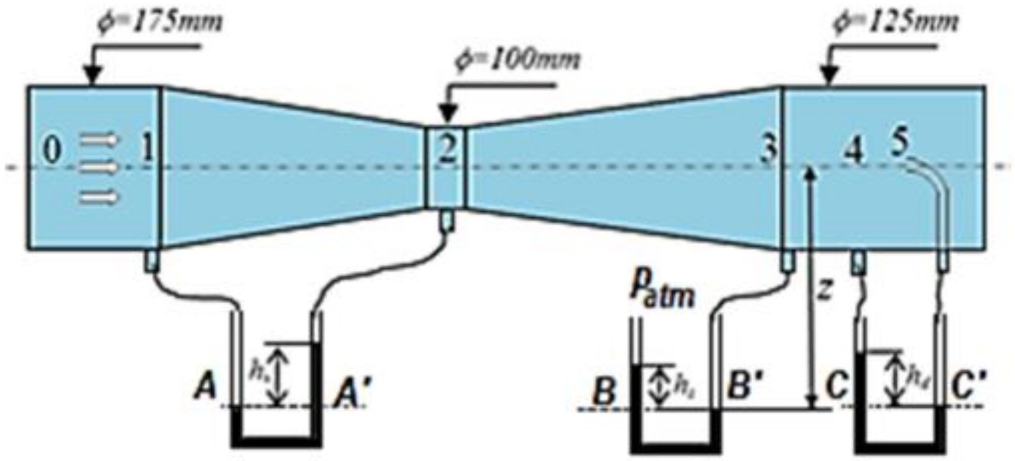
1° the fluid is perfect;

2° the fluid is incompressible;

3° the fluid is heavy;

4° the flow is steady.

Given: $P_0 = \frac{1.5x10^5 N}{m^2}$, $P_{atm} = \frac{10^5 N}{m^2}$, z=1m



Determine:

1. The volumetric flow rate Q_V in the pipe, given that the pressure head difference h_v=40 mmHg.
2. The flow velocities.
3. The pressure P_2.
4. The heads: $h_s and\ h_d$.

Solution N°8:

1- Volume flow Qv

$$Q_V = A_1V_1 = A_2V_2$$

Where:

$V_1 = \frac{A_2}{A_1} V_2$ and $V_2 = \frac{A_1}{A_2} V_1$

To obtain v_1, we apply Bernoulli's equation between points (1) and (2):

$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$ with $z_1 = z_2$

We'll have: $\frac{P_1 - P_2}{\rho} = \frac{V_2^2 - V_1^2}{2}$

And from hydraustatic : $P_1 - P_2 = \rho_{Hg}.g.h_v$

So:

$P_1 - P_2 = \frac{\rho_{H2O}}{2}(V_2^2 - V_1^2) = \frac{\rho_{H2O}}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) Q_V^2 = \rho_{Hg}.g.h_v$

Therfore:

$$Q_V = \sqrt{\frac{\rho_{Hg}.g.h_v}{2\rho_{H2O}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)}}$$

$$Q_V = \sqrt{\frac{\rho_{Hg}.g.h_v}{2\rho_{H2O}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)}} = \sqrt{\frac{13600x9.81x0.04}{2x1000\left(\frac{1}{\pi(0.1)^2} - \frac{1}{\pi(0.175)^2}\right)}} = 0.1763\,\frac{m^3}{s}$$

2- calculation of speeds

$$Q_V = A_1V_1 = A_2V_2 = A_3V_3 = A_4V_4$$

$$V_1 = \frac{Q_V}{A_1} = \frac{0.1763}{\frac{3.14(0.175)^2}{4}} = 7.33\,\frac{m}{s}$$

$$V_2 = \frac{Q_V}{A_2} = \frac{0.1763}{\frac{3.14(0.1)^2}{4}} = 22.45\,\frac{m}{s}$$

$$V_3 = \frac{Q_V}{A_3} = \frac{0.1763}{\frac{3.14(0.125)^2}{4}} = 14.37\,\frac{m}{s}$$

$$V_4 = \frac{Q_V}{A_4} = \frac{0.1763}{\frac{3.14(0.125)^2}{4}} = 14.37\,\frac{m}{s}$$

Point (5) is a stagnation point; therefore, $V_4 = 0$

3- Calculation of P_2

$P_1 - P_2 = \rho_{Hg}.g.h_v$ with $P_1 = P_0$

So, $P_2 = P_0 - \rho_{Hg}.g.h_v$

$$P_2 = 10^5 - 13600x9.81(0.04) = 94663.36\,Pa$$

$P_{B'} = P_B = P_{atm} + \rho_{Hg}.g.h_S$ and $P_{B'} = P_B = P_{atm} + \rho_{Hg}.g.h_S = \frac{P_B - P_{atm}}{\rho_{Hg}.g}$

$$P_{B'} = P_3 + \rho_{H2O}.g.z_3$$

P_3: can be determined by applying Bernoulli's equation between points (2) and (3).

$\frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 = \frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3$ with $z_2 = z_2$

$P_3 = P_2 + \frac{\rho(V_2^2 - V_3^2)}{2}$; $P_3 = 94663.36 + \frac{1000(22.45^2 - 14.37^2)}{2} = 243413.36\,Pa$

So;

$$P_{B'} = P_3 + \rho_{H2O}.g.z_3 = 243413.36 + 1000x9.81x1 = 253223.36\,Pa$$

And from $P_{B'} = P_B = P_{atm} + \rho_{Hg}.g.h_S$

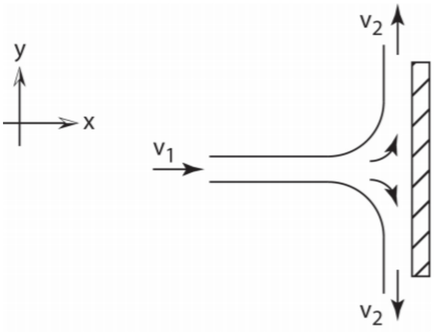
$$h_S = \frac{P_B - P_{atm}}{\rho_{Hg}.g} = \frac{253223.36 - 100000}{13600x9.81} = 1.14\,m$$

The height h_D:

$$h_D = \frac{\frac{1}{2}\rho V_4^2}{g(\rho_{Hg} - \rho_{H2O})} = \frac{0.5x1000x14.37^2}{9.81(13600 - 1000)} = 0.835\,m$$

Exercise N°9:

A water jet of 25 mm diameter strikes a flat plate perpendicular to the jet axis. The force exerted on the plate is 650 N. Determine the volumetric flow rate in m³/s.



Solution N°9:

For a water jet striking a flat plate perpendicularly, the force F_{Tx} is related to the flow rate Q_V, velocity V_{X1}, and water density ρ :

$$F_{Tx} = \rho Q_V(0 - V_{X1})$$

Also $Q_V = A_1.V_1$, where A is the jet cross-sectional area

$$Q_V = V_1 \frac{\pi D_{jet}^2}{4}$$

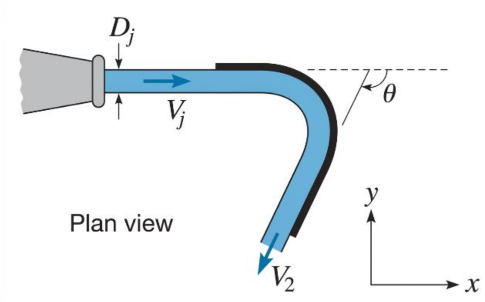
$$650 = -1000xA_1xV_1^2 = -1000\frac{Q_V^2}{A_1}$$

Finally we find : $Q_V = 0.018\frac{m^3}{s}$

Exercise N°10:

In the figure below, assume that friction is negligible, $\theta = 150°$, and that the water jet has a velocity of 29 m/s and a diameter of 2.54 cm. Determine:

1. The component of the force acting on the blade in the direction of the jet;
2. The component of the force acting normal to the jet;
3. the magnitude and direction of the resultant force exerted on the blade.



Solution N°10:

1. The component of the force acting on the blade in the direction of the jet

$$F_x = \dot{m}\Delta V = \rho . A . V_1(V_1 - V_2 cos\theta) = \frac{\pi D^2}{4} = \frac{\pi(0.0254)^2}{4} = 5.06\ 10^{-4} m^2$$

$$F_x = \rho . A . V_1(V_1 - V_2 cos\theta) = 1000x5x10^{-4}x29(29 - 29cos115) = 245.63\ N$$

Therefore, the force acting on the blade is equal and opposite to

$$F_x = 245.63\ N$$

2. The component of the force acting normal to the jet;

$$F_y = \dot{m}V_2 sin\theta = \rho x A x V_2 sin\theta$$

$$F_y = 1000x0.0005x29sin115 = 13.14\ N$$

Therefore, the force acting on the blade is equal and opposite to

$$F_y = 13.14\ N$$

3. The resulting force is :

$F = \sqrt{F_x^2 + F_y^2} = \sqrt{245.63^2 + 13.14^2}$=245.98 N

Exercise N°11:

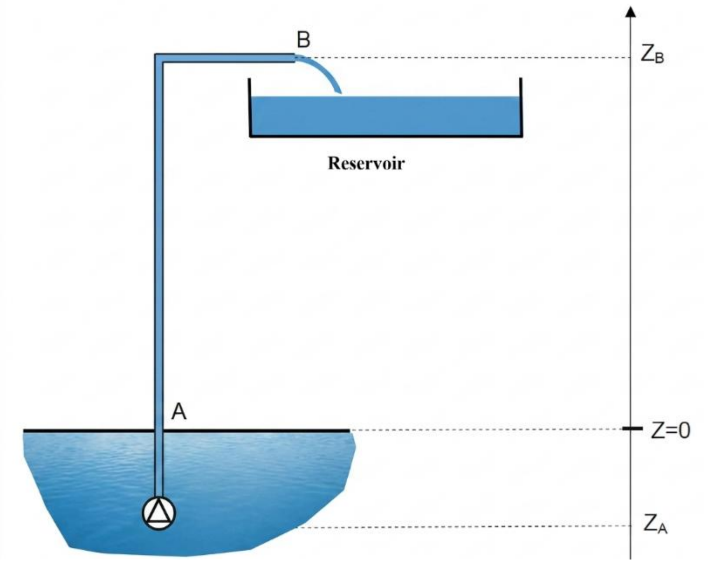
A reservoir is to be filled by pumping water from the groundwater table. For this purpose, a submersible pump is used to draw water from point A, located at an elevation of ZA=−26 m. The pressure at point A is PA=2 bar.

The water delivered by the pump flows through a circular pipe with an internal diameter of d=31mm. The water is discharged at point B, located at an elevation of ZB=30 m, with a volumetric flow rate of qv=2772 L/h. The pressure at point B is assumed to be PB=1 bar.

The pump is driven by an electric motor. The overall efficiency of the motor–pump system is η=80%.

Assume that the fluid is perfect and the suction velocity is equal to the discharge velocity (VA=VB=V)

1. Calculate the mass flow rate q_m of the pump.
2. Determine the water flow velocity V.
3. Using Bernoulli's theorem, determine the net power P_n supplied by the pump.
4. Calculate the electrical power consumption P_e.



Solution N°11:

1. the mass flow rate q_m:

$$q_m = \rho q_V$$

$$q_m = 1000\frac{2772x10^{-3}}{3600} = 0.77\frac{kg}{s}$$

2. water flow velocity V

$q_V = V.A; V = \frac{4q_V}{\pi d^2}$

$$V = \frac{4\frac{2772x10^{-3}}{3600}}{\pi x 0.031^2} = 1.02\, m/s$$

3. Use the Bernoulli equation to analyze the flow between A and B we obtain:

$$\frac{V_B^2 - V_A^2}{2} + \frac{P_B - P_A}{\rho} + g(Z_B - Z_A) = \frac{P_n}{q_m}$$

Since $V_B = V_A$ then :

$$P_n = q_m\left(\frac{P_B - P_A}{\rho} + g(Z_B - Z_A)\right)$$

$$P_n = 0.77\left(\frac{1.10^5 - 2.10^5}{1000} + 9.81x(30 - -26)\right) = 346\,watt$$

4. Electrical power consumption P_e

The pump efficiency is expressed as:

$\eta = \frac{P_n}{P_e} \Longrightarrow \quad P_e = \frac{P_e}{\eta}$

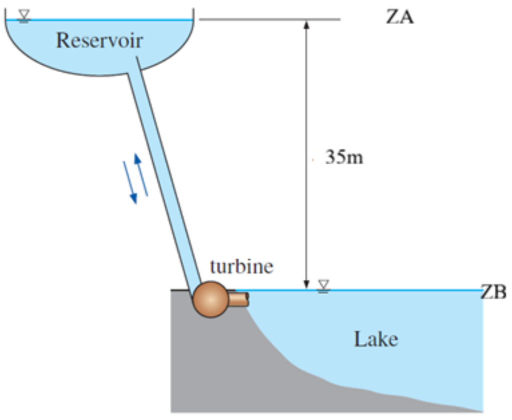
$$P_e = \frac{346}{0.8} = 432\,Watt$$

Exercise N°12:

A pipe is connected to the outlet of a turbine in order to discharge water into a lake. The free surface level of the lake, denoted ZB is assumed to remain constant.

The mass flow rate passing through the turbine is Qm=175 kg/s. The elevation difference between the upstream point and the lake surface is H=ZA−ZB=35 m.

1. Using Bernoulli's theorem, determine the useful power P_U developed by the turbine. Clearly state all simplifying assumptions.
2. Determine the shaft power recovered from the turbine if its overall efficiency is η=70%.



Solution N°12:

1. We apply Bernoulli's theorem between points A and B, and we obtain:

$$\frac{V_B^2 - V_A^2}{2} + \frac{P_B - P_A}{\rho} + g(Z_B - Z_A) = \frac{P_U}{q_m}$$

Since $P_B = P_A = P_{atm}$ and $V_B = V_A = 0$ then :

$$P_U = q_m\, g(Z_B - Z_A) = q_m\, gH = 175x9.81x35 = 60025\, watt$$

2. Shaft power recovered from the turbine:

$$P_a = \eta P_U = 60025x0.7 = 45018\, watt$$

Chapter 4 : Real incompressible fluid dynamics

Chapter Objectives

By the end of this chapter, the student should be able to:

- ✓ Evaluate The Reynolds Number;
- ✓ Identify The Different Flow Regimes Of A Fluid (Laminar, Transitional, And Turbulent);
- ✓ Calculate Both Minor (Singular) And Major (Distributed) Head Losses;
- ✓ Apply Bernoulli's Theorem To Steady Flow Of A Real Incompressible Fluid.

1 Introduction:

In the previous chapter, the fluid was assumed to be ideal in order to apply the energy conservation equation in a simplified form. However, the flow of a real viscous fluid is significantly more complex than that of an ideal fluid. In practice, frictional forces arise due to the fluid's viscosity. These forces act both between adjacent fluid layers and between the fluid and the solid boundaries of the conduit.

To analyze and solve real fluid flow problems, experimental results are required, particularly those established by the British engineer and physicist Osborne Reynolds, who provided fundamental insights into flow behavior and regime classification.

Based on these experimental findings, simplified approaches for evaluating head losses in pipe flows have been developed. These methods are essential for the design and sizing of hydraulic systems such as pumping installations, turbines, and hydraulic or thermal machines in which real fluids are transported.

2 Real fluid:

A real fluid is defined by the presence of internal frictional forces during motion. Unlike an ideal fluid, the contact forces exerted on a surface element are not strictly perpendicular; they include tangential components (shear forces) that resist the sliding of fluid layers over one another. This inherent resistance to flow, caused by molecular cohesion, is characterized by a physical property known as viscosity.

3 Flow regimes - Reynold's Experiment

Understanding the flow regime of a fluid is essential for the analysis and design of hydraulic installations and fluid flow systems. Indeed, the flow regime affects many physical phenomena, particularly heat and mass transfer, as well as head losses.

In 1883, the British engineer and physicist Osborne Reynolds (1842–1912), Professor of Engineering at the University of Manchester, conducted a series of landmark experiments on fluid flow through a straight cylindrical pipe. By injecting a dye along the centreline of the pipe, he observed the following:

- At low flow velocity, the dye streak remained tightly concentrated along the axis, indicating a stable laminar flow regime in which fluid particles travel along ordered, parallel paths (figure a);

- At higher flow velocity, vortical structures began to develop and intensify, causing the dye to rapidly spread across the entire cross-section of the pipe — a characteristic signature of turbulent flow (figures b and c). This turbulent diffusion was found to far exceed molecular diffusion, which remains virtually imperceptible under laminar conditions.

By systematically varying the flow rate, the pipe diameter, and the nature of the fluid — and therefore its viscosity — Reynolds demonstrated that the transition between laminar and turbulent regimes is entirely governed by a single dimensionless parameter, now universally known as the Reynolds number, denoted Re. This number expresses the ratio of inertial forces to viscous forces within the flow, and is defined by the following relation:

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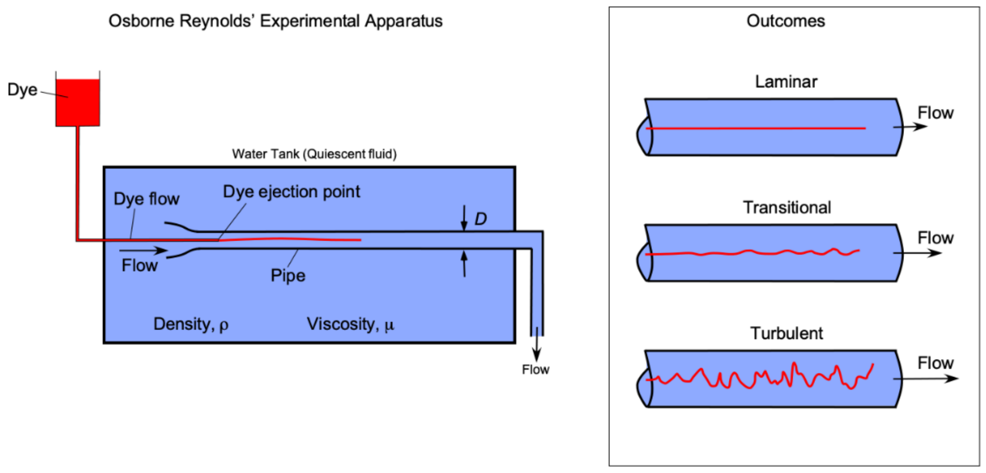


Figure 4. 1 Reynolds experiment

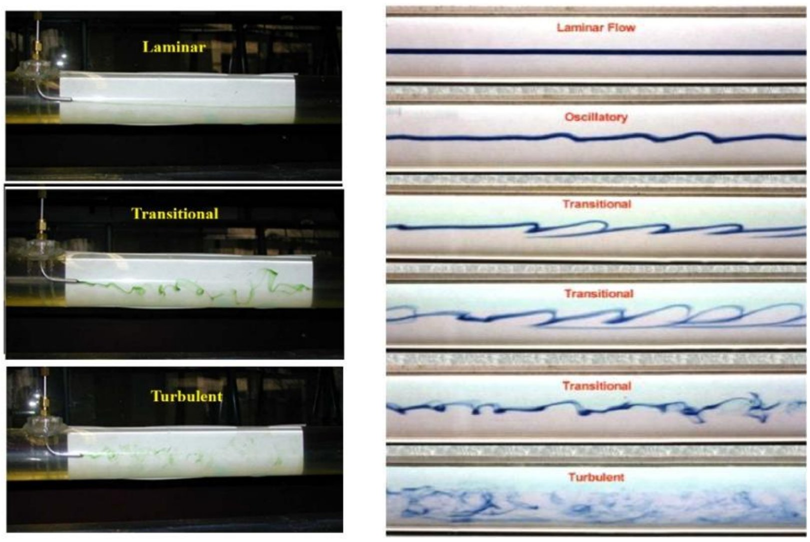


Figure 4. 2 Visualization of flow regimes in Reynolds experiment

$$Re = \frac{\rho V D}{\mu}$$

ρ: density V: flow velocity D: diameter μ: dynamic viscosity

Indicative empirical results:

If $Re \leq 2300$ laminar flow

If $2300 \leq Re \leq 4000$ Transitional flow

If $Re < 10^5$ Smooth turbulent flow

If $Re > 10^5$ Rough turbulent flow

These limits may vary slightly depending on the reference source and flow conditions. However, in practice, Reynolds number values are usually sufficiently far from these thresholds to clearly identify the flow regime.

In turbulent flows, the Reynolds number can reach very high values, typically ranging from 10^5 to 10^8.

Hydraulic Diameter

When a pipe or channel does not have a circular cross-section, an equivalent diameter called the hydraulic diameter is defined. This parameter makes it possible to analyze flow in non-circular conduits using the same relations as those used for circular pipes.

The hydraulic diameter D_h is defined as:

$$D_h = \frac{A}{P}$$

A: is the flow cross-sectional area;[m²]

P : is the wetted perimeter, i.e., the portion of the wall in contact with the fluid [m].

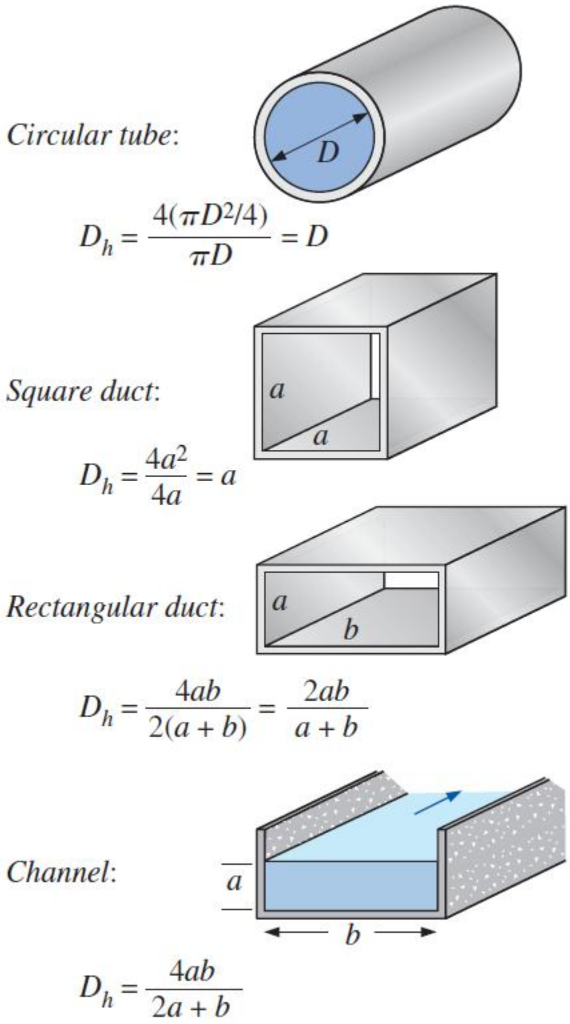


Figure 4. 3 Hydraulic diameter for some geometric shapes

4 Dimensional Analysis

Physical phenomena are often so complex that they cannot be described by fully solvable mathematical equations, given the large number of parameters involved. To overcome this difficulty, dimensional analysis is employed. This technique groups the various physical quantities into dimensionless numbers, making it possible to study and compare the influence of each parameter on the phenomenon under consideration, thereby greatly simplifying the problem.

Experience shows that some of these dimensionless numbers remain constant when the scale is changed. It is therefore possible to extrapolate results obtained from laboratory experiments to industrial-scale installations — this is the principle of similarity (or similitude).

Dimensional analysis is widely used in physics, chemistry, and engineering. In particular, it allows:

- ✓ a priori verification of the dimensional consistency of an equation or a computed result;
- ✓ formulation of simplifying hypotheses about the governing quantities of a physical system;
- ✓ preparation for a more complete theory that will confirm or refute those hypotheses.

5 Concepts of Dimensions

5.1 Systems of Units

For any physical quantity, units are the fundamental tool of quantification. A system of units is a coherent set of units that enables the definition of all observable quantities. Two major systems are distinguished: the metric (or international) system and the imperial system, still widely used in Anglo-Saxon countries.

Within the metric system, physicists favour the MKS system (Metre – Kilogram – Second), while chemists traditionally use the CGS system (Centimetre – Gram – Second). Although conversions between systems may appear straightforward, they require careful attention when dealing with empirical relations. Reference handbooks provide detailed conversion tables for this purpose.

5.2 Dimension

The dimension of a physical quantity expresses the relationship between its unit and the seven base units of the International System (S.I.). These seven fundamental units are summarised in the table below:

Table 4. 1 SI Base Units and Dimensions

Dimension	Symbol	S.I. Unit
Mass	M	kilogram (kg)
Time	T	second (s)
Length	L	metre (m)
Temperature	θ	Kelvin (K)
Electric current	I	Ampere (A)
Amount of substance	n	mole (mol)
Luminous intensity	J	Candela (cd)

5.3 The Vaschy-Buckingham Theorem

Dimensional analysis rests on a fundamental principle: only quantities of the same dimension can be compared. For example, two lengths may be compared, but a length and a mass cannot. This principle is formalised mathematically by the Vaschy-Buckingham theorem (also known as the π theorem).

5.3.1 Statement of the Theorem

Consider a physical problem involving N distinct quantities (variables) $(A_1, A_2, A_3, \ldots, A_n)$, such as velocity, pressure, viscosity, etc., whose fundamental dimensions number J. There exists a functional relationship linking all these quantities, written in the general form:

$$f(A_1, A_2, A_3, \ldots, A_n) = 0$$

If $\pi_1, \pi_2, \pi_3, \ldots$ denote the dimensionless groups formed from the physical quantities $A_1, A_2, \ldots, A_n$, the preceding relation can be rewritten in the reduced form:

$$\Phi(\pi_1, \pi_2, \pi_3, \ldots, \pi_k) = 0 \quad \text{with } K < N$$

The number K of dimensionless π groups equals the number of initial physical variables N minus the number of fundamental dimensions J involved (usually $J = 3$ for mechanical problems):

$$K = N - J$$

5.3.2 Application Procedure — Nine Steps

To carry out a dimensional analysis, the following nine steps are applied:

1. List all physical quantities (A_i) involved in the problem along with their dimensions. Omit any quantity that depends on another quantity already listed.
2. Write the general functional relationship:

$$f(A_1, A_2, A_3, \ldots, A_n) = 0$$

3. Select the repeating variables (J variables). These must collectively span all fundamental dimensions of the problem. Typically, one variable is chosen for the geometric scale, one for kinematic conditions, and one for forces or mass.
4. Express each π parameter in terms of the repeating variables raised to unknown exponents. For example:

$$\pi_1 = v^{x_1} \cdot d^{y_1} \cdot \rho^{z_1} \cdot \mu = (L/T)^{x_1} \cdot L^{y_1} \cdot (M/L^3)^{z_1} \cdot (M/LT) = M^0 \cdot L^0 \cdot T^0$$

Ensure that all quantities A_i are included in the π_i groups.

5. For each fundamental dimension, write the algebraic equation of exponents, requiring the sum to be zero (dimensional homogeneity condition).
6. Solve the resulting system of equations.
7. Substitute the found exponents (x_1, y_1, z_1, …) back into the π expressions (from Step 4) to obtain the dimensionless groups.
8. Determine the final dimensionless function:

$$\Phi(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-j}) = 0$$

9. Verify that all π parameters are mutually independent.
10. Identify and reformulate the obtained dimensionless groups as recognised dimensionless numbers (Re, Ma, Fr, We, etc.).

5.4 Example of Dimensional Analysis: Reynolds Number

The Reynolds number Re is influenced by the physical properties of a fluid and the flow conditions. It is assumed to depend on the fluid density ρ, dynamic viscosity μ, characteristic velocity V, and characteristic length L.

Using dimensional analysis, derive the dimensionless expression of the Reynolds number.

Re=f(ρ,μ,V,L).

Solution

List of parameters and their dimensions

1. Reynolds number: Re→ dimensionless
2. Density: ρ $[ML^{-3}]$
3. Velocity: V $[LT^{-1}]$
4. Characteristic length: L [L]
5. Dynamic viscosity: μ $[ML^{-1}T^{-1}]$

Assume that the Reynolds number can be expressed as:

$$Re = f(\rho, \mu, V, L)$$

$$Re = K.\rho^{a} V^{b} L^{c} \mu^{d}$$

where K is a dimensionless constant determined experimentally.

Replacing each variable by its fundamental dimensions gives:

$$1 = M^0.L^0.T^0 = (M^a.L^{-3a}).(M^b.L^{-b}.T^{-b}).(L^c.T^{-c}).L^d$$

$$M^0.L^0.T^0 = M^{a+b}.L^{-3a-b+c+d}.T^{-b-c}$$

By equating the exponents of the fundamental dimensions M, L, and T, we obtain:

$$\begin{cases} a + d = 0 \\ -3a - b + c + d = 0 \\ -b - c = 0 \end{cases} \Rightarrow \begin{cases} a = -b \\ d = -b \\ c = -b \end{cases}$$

By substituting in Re we find :

$$Re = K.\rho^{-b} V^{-b} L^{-b} \mu^{b} = K \left(\frac{V.L.\rho}{\mu}\right)^{-b}$$

From physical analysis and experimental validation, the constant K is generally taken as: K=1

Therefore, the final expression of the Reynolds number is:

$$Re = \frac{V.L.\rho}{\mu}$$

6 Head losses

Head losses refer to the reduction in the mechanical energy of a fluid as it flows through pipes or hydraulic systems.

This energy loss is mainly caused by:

- friction between the fluid and the pipe walls;
- turbulence generated by changes in flow direction or cross-sectional area;
- hydraulic fittings such as bends, valves, elbows, and joints.

Head losses generally appear as a pressure drop or a decrease in hydraulic head along the flow path.

Two types of head losses are commonly distinguished:

1. **Major head losses**: caused by friction along straight pipes.
2. **Minor head losses**: caused by fittings and local disturbances.

Major losses / Friction losses

6.1 Concept of Pipe Roughness

Unlike a smooth surface, a rough pipe surface contains irregularities that directly affect the friction forces between the fluid and the pipe wall.

The internal surface of a rough pipe may be considered as a series of small protrusions or asperities. The average height of these irregularities is denoted by ε\varepsilonε and is called the absolute roughness.

Pipe roughness plays an important role in fluid flow because it increases head losses and affects the friction factor, especially under turbulent flow conditions.

To compare the roughness with the pipe diameter, the relative roughness is defined as: $\frac{\varepsilon}{D}$

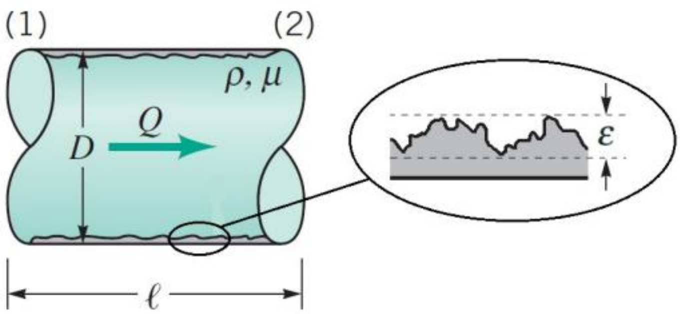


Figure 4. 4 Pipe roughness

Table 4. 2 Roughness for various typical materials

Material	**Roughness (mm)**
Riveted Steel	0.9 to 9.0
Concrete	0.3 to 3.0
Wood	0.18 to 0.9
Cast Iron	0.26
Galvanized Iron	0.15
Commercial Steel or Wrought Iron	0.045
Drawn Tubing	0.0015
Plastic, Glass	0.0 (smooth)

6.2 Major head losses

Major head losses (also known as friction losses) are generated by viscous friction along straight pipe sections. They depend on the following parameters:

- ✓ The flow regime and the internal surface roughness of the pipe ($\frac{\varepsilon}{D}$).
- ✓ The pipe diameter (1/D).
- ✓ The dynamic pressure of the fluid ($\frac{\rho v^2}{2}$).
- ✓ The length of the pipe (L).

Major head losses are directly proportional to the pipe length L and the square of the mean flow velocity V, while being inversely proportional to the pipe diameter D.

Consequently, between two points separated by a length L in a pipe of diameter D, a pressure occurs. This can be expressed in the following two forms:

$\Delta P = \lambda \frac{L}{d}\frac{\rho v^2}{2}$ or $\Delta h = \lambda \frac{L}{d}\frac{v^2}{2g}$

Where λ is a dimensionless parameter known as the friction factor. The accurate estimation of major head losses depends entirely on the determination of this coefficient. The value of λ is governed by both the flow regime and the relative roughness of the pipe's internal surface."

Laminar Flow Case (Re < 2300)

In the laminar regime, the friction factor λ is governed solely by viscous forces. Due to the very low flow velocities, the surface condition of the pipe has no impact; therefore, λ is strictly a function of the Reynolds number Re. The relationship is expressed as follows:

$$\lambda = \frac{64}{Re}$$

Turbulent Flow Case (Re > 4000)

In the turbulent regime, the pipe's surface condition becomes a critical factor, and its influence grows as the Reynolds number increases. Research has established the impact of roughness, leading to the determination of λ as a function of both the Reynolds number Re and the relative roughness $(\frac{\varepsilon}{D})$.

Several empirical formulas have been proposed, but the **Colebrook equation** is currently considered the most accurate representation of turbulent flow phenomena. It is presented in the following form:

$$\frac{1}{\sqrt{\lambda}} = -2log\,\log_{10}\left(\frac{\frac{\varepsilon}{D}}{3.7} + \frac{2.51}{Re\sqrt{\lambda}}\right)$$

6.3 Minor Losses

In addition to friction along straight sections, a fluid flowing through a piping system encounters various disruptions caused by sudden changes in geometry or direction. These are known as minor losses (or singular losses). While they are present in every installation, their cumulative effect can induce mechanical issues, such as corrosion, or hydrodynamic disturbances.

Minor losses are typically induced by components that interrupt the smooth flow of the fluid, leading to flow separation and localized mixing. Common sources include:

- Changes in cross-section: Sudden or gradual expansions (divergents) and contractions (convergents).
- Changes in direction: Bends and elbows.

- Fittings and junctions: Tees, branching connections, and valves.
- Instrumentation: Devices for flow measurement and control

Minor head losses are directly proportional to the kinetic energy of the fluid (the square of the mean velocity). They can be expressed in terms of pressure drop (ΔP) or head loss (Δh) using the following forms:

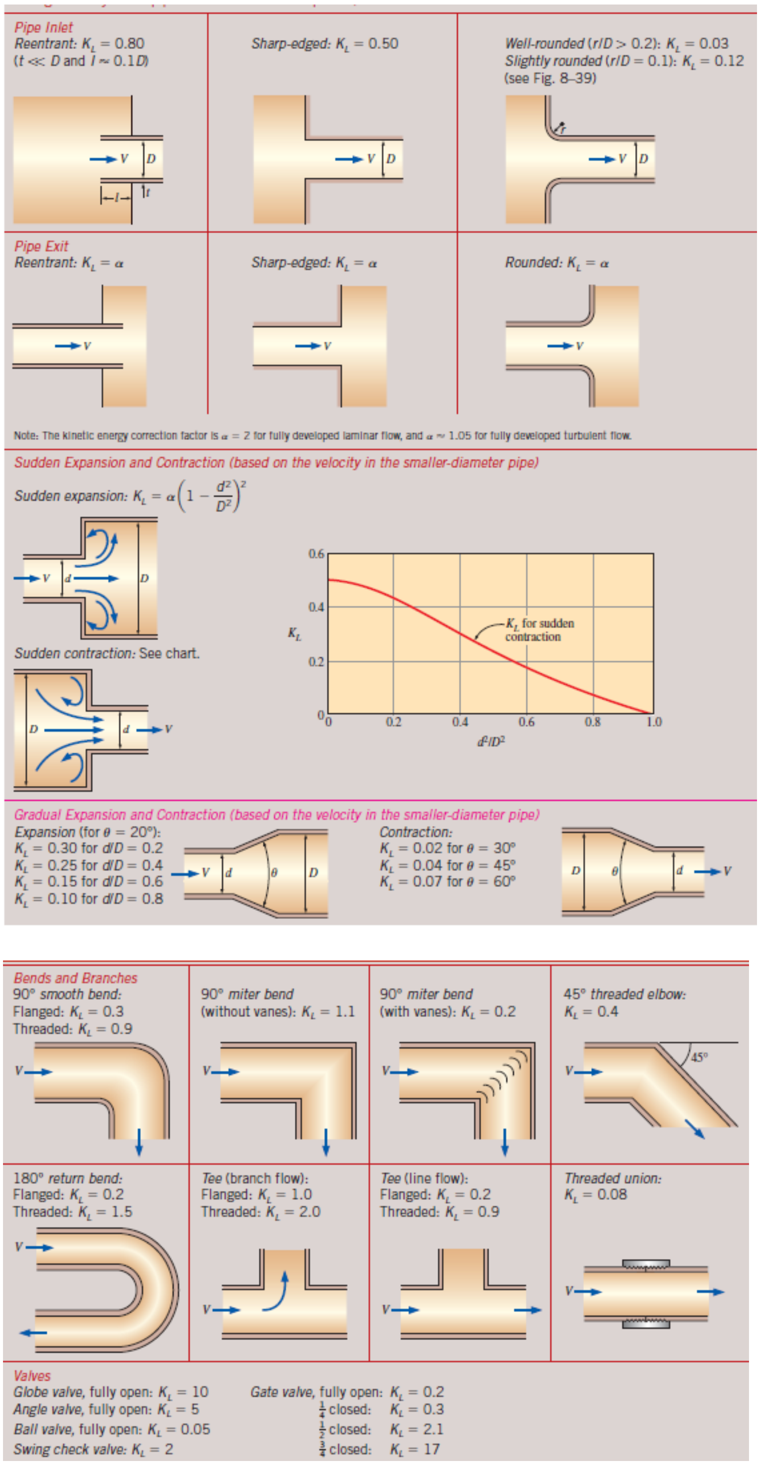
$\Delta P = K\frac{\rho V^2}{2}$ or $\Delta h = K\frac{V^2}{2g}$

Where:

- ✓ K is the minor loss coefficient. This is a dimensionless parameter that depends on the specific geometry and nature of the fitting
- ✓ V is the mean flow velocity.
- ✓ g is the acceleration due to gravity.

The values for the coefficient K are generally provided by manufacturers in their technical catalogs or can be found in engineering handbooks. Typical average values for common fittings are presented in the table below:

*Table 4. 3 Loss coefficients KLKL for various pipe fittings in turbulent flow (used in the equation hL=KLV22ghL=KL2gV2, where VV is the average velocity in the pipe containing the component)**



6.4 Moody Diagram

The Moody diagram provides a direct graphical method to determine the Darcy friction factor (*f*) for pipe flow, bypassing the iterative solution required by the implicit Colebrook equation. Using the Colebrook equation directly involves successive approximations: assume an initial *f*, compute the right-hand side, derive a new *f* from the left-hand side, and repeat until convergence (typically after 3–4 iterations).

The Moody chart, Figure 4. 5, plots the friction factor against the Reynolds number (Re) for various relative roughness values (($\frac{\varepsilon}{D}$)). The straight black line represents laminar flow, while curved blue lines depict fully turbulent flow. Developed by L.F. Moody in 1944 from Colebrook equation data—sometimes called the Moody-Stanton diagram—this tool allows users to read *f* directly once Re and ($\frac{\varepsilon}{D}$) are known, enabling accurate pressure drop calculations for both laminar and turbulent regimes.

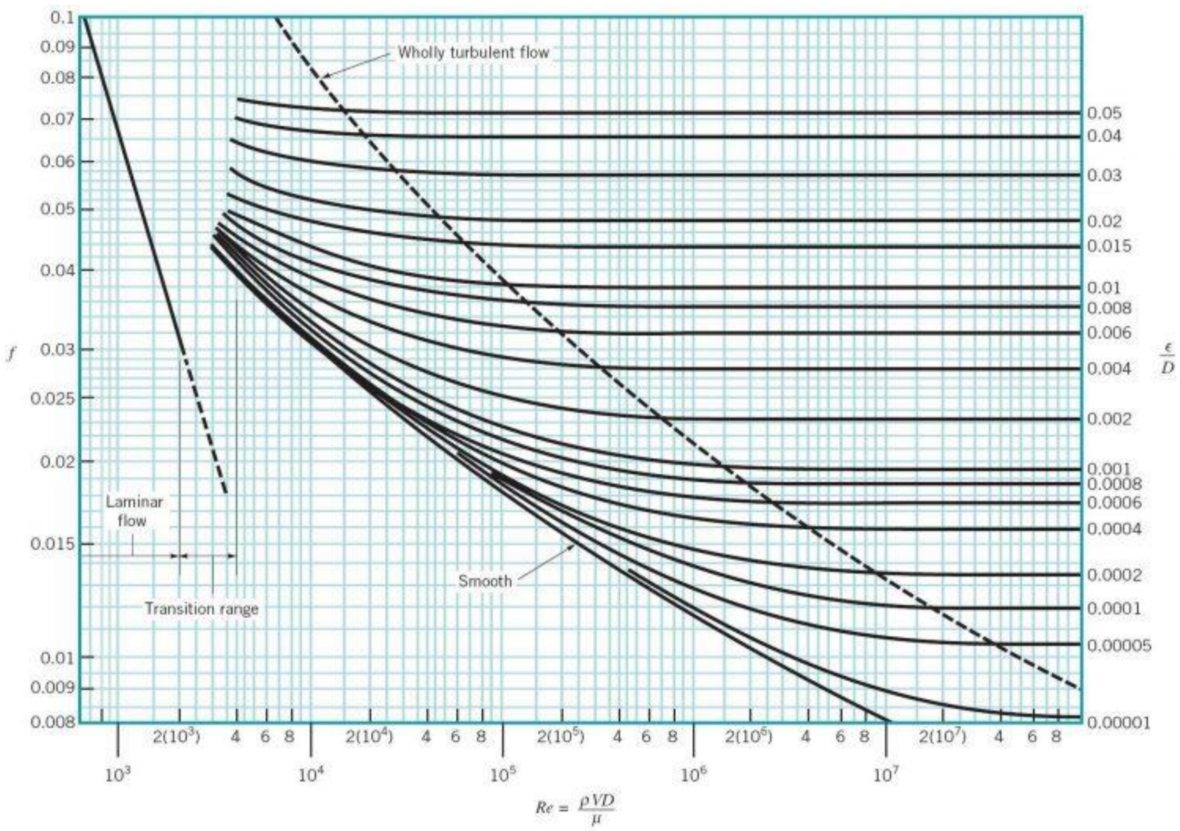


Figure 4. 5 Moody Diagram

7 Bernoulli equation for real fluids

We have seen that in the case of a real, steady-state fluid, other forces come into play, in particular frictional forces, which cause mechanical energy to dissipate into thermal energy. This phenomenon is known as frictional head loss in a fluid.

Let us reconsider the fluid streamtube diagram presented in Section 5 of Chapter 3, using the same notation and under the following assumptions:

- The fluid is real and incompressible: this implies the existence of elementary viscous friction forces dτd\taudτ, which contribute to the energy balance equation through a negative work term, leading to head losses.

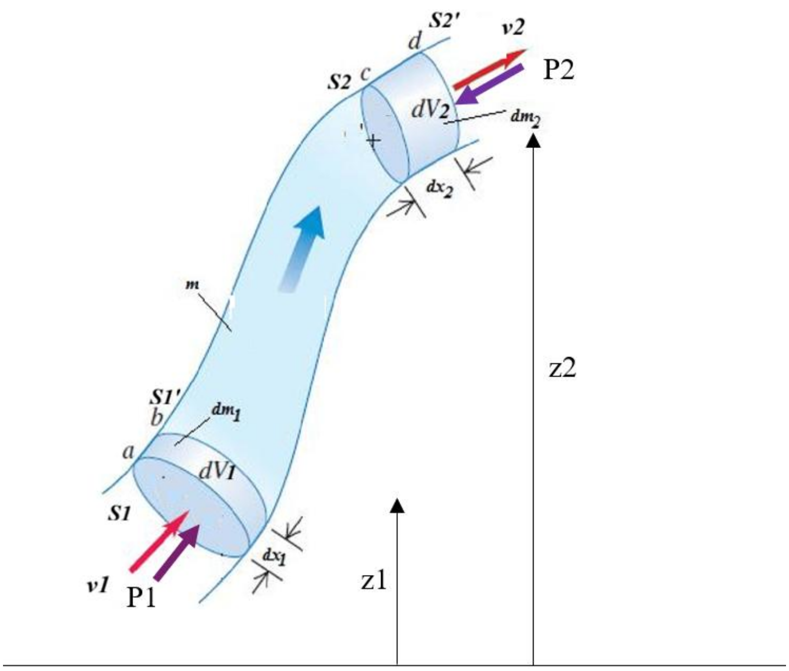


Table 4. 4 Force system on a fluid element

Consider a steady flow of an incompressible and ideal fluid through a pipe (Figure 3. 2). Between two sections S1 and S2, an elementary fluid mass dm moves along the streamline. The elevations of the centers of gravity are denoted by z1 and z2, while the flow velocities are v1 and v2. The pressures acting on the two sections are P1 and P2, respectively.

Applying the kinetic energy theorem to the fluid element, This time, the work of viscous friction forces will be taken into account. The variation of kinetic energy is equal to the sum of the external forces work:

$$\Delta E_C = \sum W$$

The change in kinetic energy between the two sections is expressed as:

$$\Delta E_C = \frac{1}{2} dm(v_2^2 - v_1^2)$$

The forces acting on the fluid are:

Gravity force: $W_P = (Z1 - Z2). g. dm$

Pressure forces: On surface $S1$: $W_{P1} = P1. S1. dx_1 = P1. dV_1$; $S1. dx_1 = dV_1$

On surface $S2$: $W_{P2} = -P2. S2. dx_2 = P2. dV_2$; $S2. dx_2 = dV_2$

On the lateral surface: $W_{Pl} = 0$

For a real fluid, internal friction forces are considered and Viscous friction work (always negative, as it opposes motion and dissipates energy) : $W_f = -\delta W_f$

$$\Delta E_C = \sum W \Longrightarrow \frac{1}{2} dm(v_2^2 - v_1^2) = P1. dV_1 - P2. dV_2 + (Z1 - Z2). g. dm - \frac{\delta W_f}{dm}$$

Dividing by dm:

$$\Longrightarrow \frac{1}{2}(v_2^2 - v_1^2) = P1. \frac{dV_1}{dm} - P2. \frac{dV_2}{dm} + (Z1 - Z2). g - \frac{\delta W_f}{dm}$$

Knowing that:

$$dm = \rho. dV_1 = \rho. dV_2$$

The equation becomes:

$$\frac{1}{2}(v_2^2 - v_1^2) = \frac{P1}{\rho} - \frac{P2}{\rho} + (Z1 - Z2). g - \frac{\delta W_f}{dm}$$

Bernoulli's equation is obtained as follows:

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{P2 - P1}{\rho} + (Z1 - Z2). g = \frac{\delta W_f}{dm}$$

The head loss between points (1) and (2) is defined by: $\frac{\delta W_f}{dm}$

which represents the energy loss due to viscous friction per unit mass of fluid passing through.

Each term in equation (4) has units of joules per kilogram (J/kg).

Dividing equation (4) by g, each term becomes homogeneous to a length in meters (m):

$$\frac{1}{2g}v_1^2+\frac{P1}{\rho g}+Z1=\frac{1}{2g}v_2^2+\frac{P2}{\rho g}+Z2+\frac{\delta W_f}{g.dm}$$

The Bernoulli equation can then be written as:

$$\frac{1}{2g}v_1^2+\frac{P1}{\rho g}+Z1=\frac{1}{2g}v_2^2+\frac{P2}{\rho g}+Z2+H_{1-2}$$

This is Daniel Bernoulli's theorem for a real fluid, which states that at every point in steady flow, the elevation head, the pressure head, the velocity head, and the head loss together form a constant sum.

H is a positive quantity, expressed in [Pa], representing the sum of all head losses — both minor (singular) and major (linear) losses — between sections (1) and (2).

It can be interpreted graphically as follows:

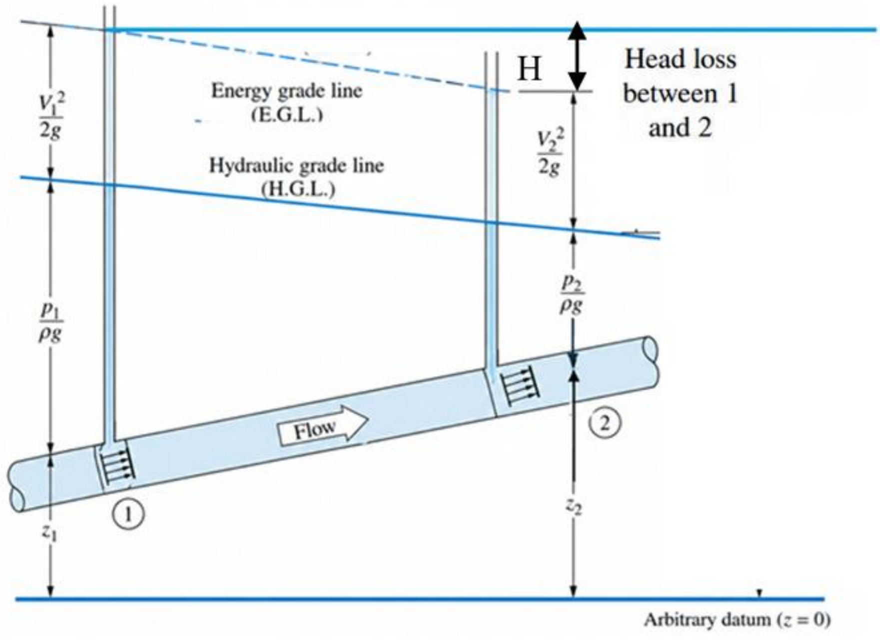


Table 4. 5 Schematic illustration of the energy equation, for an incompressible, real kflow.

8 Bernoulli equation with energy transfer

In hydraulic systems, a hydraulic machine installed in a pipe may either transfer mechanical energy to the fluid, as in the case of a pump, or extract energy from the fluid, as in the case of a turbine. Consequently, Bernoulli's equation must be modified to account for this energy exchange.

The generalized Bernoulli equation for a real incompressible fluid between two points (1) and (2) is written as:

$$\frac{1}{2g}v_1^2 + \frac{P1}{\rho g} + Z1 \pm \Delta Hp = \frac{1}{2g}v_2^2 + \frac{P2}{\rho g} + Z2 + H_{1-2}$$

where:

- $\Delta HP > 0$ for a pump (energy supplied to the fluid),
- $\Delta HP < 0$ for a turbine (energy extracted from the fluid),
- H_{1-2} represents the head losses due to viscous effects and friction.

If the energy transfer occurs from the machine walls to the fluid, the machine acts as a pump. Conversely, if the energy transfer occurs from the fluid to the machine walls, the machine acts as a turbine.

Thus, the term ΔHP represents the increase or decrease in the total mechanical energy per unit weight of the flowing fluid caused by the hydraulic machine.

PRACTICAL EXERCISES

Exercise N°1

Determine the critical velocity vc:

a) For medium fuel oil at 15°C flowing through a pipe with a diameter of 15 cm;

b) For water at 15°C flowing through the same pipe.

The kinematic viscosity at 15°C is given as: $vfuel = 4.47 \times 10 - 6\, m2/s$ for the fuel oil, and $vwater = 1.142 \times 10 - 6\, m2/s$ for water.

Solution N°1:

For a pipe flow, the upper limit of the Reynolds number for maintaining a strictly laminar flow (or the lower critical Reynolds number) is typically taken as $Re = 2000$

a) For medium fuel oil at 15°C:

Using the Reynolds number formula:

$$2000 = Re = \frac{vc \cdot d}{vfuel} = vc \cdot \frac{0.15}{4.47 \times 10 - 6}$$

Therefore, solving for vc:

$$vc = \frac{2000 \times 4.47 \times 10 - 6}{0.15} = 0.059\, m/s$$

b) For water at 15°C:

Using the same pipe diameter (d = 0.15 m) and the Reynolds number threshold:

$$2000 = Re = \frac{vc \cdot d}{vwater} = vc \cdot \frac{0.15}{1.142 \times 10 - 6}$$

Therefore, solving for vc:

$$vc = \frac{2000 \times 1.142 \times 10 - 6}{0.15} = 0.015\, m/s$$

Exercise N°2:

A water tank is being drained by a vertical pipe. The pipe has a diameter of 50 mm and a length of 73 m. Neglecting minor losses and assuming a linear friction factor coefficient of $\lambda = 0.025$, calculate the discharge flow rate.

Solution N°2:

Let us apply Bernoulli's equation between point 1 (free water surface) and point 2 (pipe exit):

$$\frac{P1}{\rho g} + \frac{V1}{2g} + Z1 = \frac{P2}{\rho g} + \frac{V2}{2g} + Z2 + \Delta H$$

By setting the elevation datum reference line at the pipe exit, we define the following boundary conditions:

- ✓ Z1 = 80 m and Z2 = 0 m
- ✓ P1 = P2 = Patm (both boundaries are exposed to atmospheric pressure)
- ✓ V1 = 0 m/s (large tank dimensions relative to the cross-sectional area of the pipe)

Substituting these terms and expressing the linear friction head losses as $\Delta H = \lambda \cdot \frac{L}{D} \cdot \frac{V2}{2g}$, the equation simplifies to:

$$\frac{V2}{2g} + \lambda \cdot \frac{L}{D} \cdot \frac{V2}{2g} = 80$$

Isolating the exit velocity V2:

$$V2 = \sqrt{\frac{2g \cdot 80}{1 + \lambda \cdot \frac{L}{D}}} = \sqrt{\frac{2 \times 9.81 \times 80}{1 + 0.025 \times \frac{73}{0.05}}} = 6.47\, m/s$$

Finally, the discharge volumetric flow rate Q is computed as the product of velocity and the cross-sectional area of the pipe:

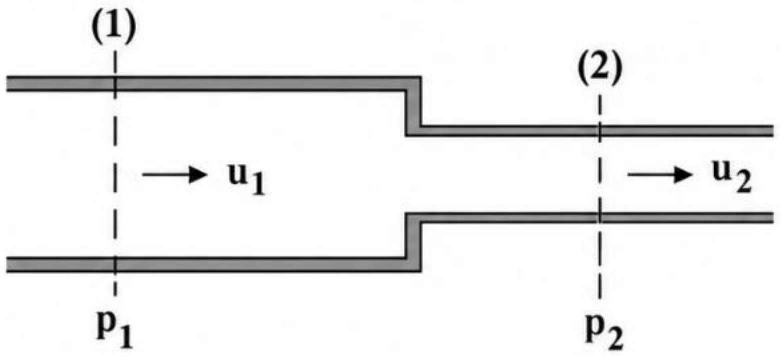
$$Q = V2 \cdot S2 = 6.47 \times \frac{3.14 \times 0.052}{4} = 0.0127\, m3/s$$

Exercise N°3:

A pipe transporting water undergoes a sudden section contraction. The cross-sectional area at section (1) is $A_1 = 0.002$ m² and at section (2) is $A_2 = 0.001$ m². The pressure at section (2) is $p_2 = 500$ kPa and the fluid velocity is $u_2 = 8$ m/s. The minor loss coefficient K for this sudden contraction is 0.4. The density of water is $\rho = 1000$ kg/m³.

Calculate the following parameters:

1. The mass flow rate ($\dot{m}$).
2. The static pressure at section (1) (p_1).
3. The total force acting on this pipe section.



Solution N°3:

1. Calculation of the Mass Flow Rate ($\dot{m}$)

By the principle of conservation of mass (continuity equation), the volumetric flow rate remains constant:

$$A1 \cdot V1 = A2 \cdot V2$$

Therefore, the upstream flow velocity V_1 is determined as:

$$V1 = V2 \cdot \frac{A2}{A1} = 8 \times \frac{0.001}{0.002} = 4\, m/s$$

The mass flow rate is given by:

$$\dot{m} = \rho \cdot A1 \cdot V1 = 1000 \times 0.002 \times 4 = 8\, kg/s$$

The corresponding volumetric flow rate is:

$$QV = A1 \cdot V1 = 0.002 \times 4 = 0.008\, m3/s$$

2. Calculation of the Upstream Pressure (p_1)

First, we compute the localized pressure head loss caused by the sudden section contraction using the minor loss coefficient K (referenced here to the upstream velocity V_1):

$$\Delta p = \frac{1}{2} \cdot \rho \cdot K \cdot V1 = \frac{1}{2} \times 1000 \times 0.4 \times 42 = 3200\, Pa$$

Now, we apply the extended Bernoulli equation accounting for head losses between sections (1) and (2) with $z_1 = z_2$ (horizontal flow pipe):

$$\frac{p1}{\rho g} + \frac{V1}{2g} = \frac{p2}{\rho g} + \frac{V2}{2g} + \frac{\Delta p}{\rho g}$$

Isolating p_1 yields:

$$p1 = p2 + \frac{\rho \cdot (V2 - V1)}{2} + \Delta p$$

Substituting the numerical values into the equation:

$$p1 = 500 \times 103 + \frac{1000 \cdot (82 - 42)}{2} + 3200$$

$$p1 = 500{,}000 + 24{,}000 + 3{,}200 = 527.2\, kPa$$

3. Calculation of the Total Force Acting on the Section (F)

Applying the momentum conservation equation to the fluid control volume between sections (1) and (2):

$$F = (p1 \cdot A1 + \dot{m} \cdot V1) - (p2 \cdot A2 + \dot{m} \cdot V2)$$

Substituting the calculated values:

$$F = [(527.2 \times 103 \times 0.002) + (8 \times 4)] - [(500 \times 103 \times 0.001) + (8 \times 8)]$$

$$F = [1054.4 + 32] - [500 + 64] = 1086.4 - 564 = 522.4\, N$$

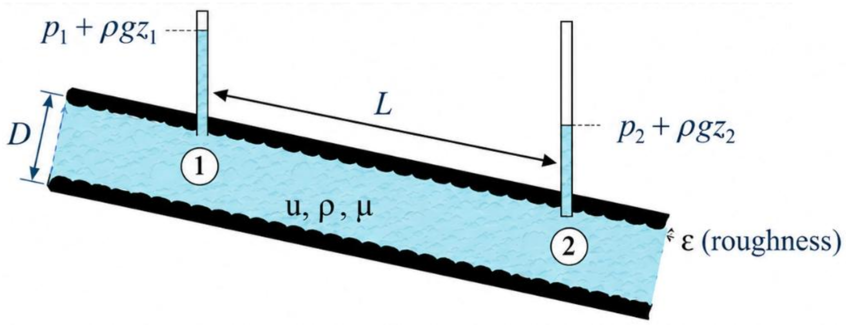
Exercise N° 4:

The figure below illustrates the steady, fully-developed flow of a viscous fluid through a rough cylindrical pipe. The pressure loss per unit length ($\Delta p/L$) between sections (1) and (2) depends on the following physical quantities:

- D — inner diameter of the pipe (m)
- ρ — fluid density (kg/m³)
- μ — dynamic viscosity of the fluid (Pa·s)
- U — mean flow velocity (m/s)
- ε — absolute wall roughness (m)

The objective is to carry out a dimensional analysis using the Buckingham π theorem, in order to express the dimensionless pressure loss as a function of the relevant π groups governing this problem.

1. Identify all the variables involved in this problem.
2. Express all variables in terms of their fundamental dimensions (M, L, T).
3. Apply the Buckingham π theorem to determine the dimensionless groups of the problem.
4. Write the final relation in terms of the π groups.



Solution N°4:

1. Variables Involved

The variables involved in this problem are: Δp/L, D, ρ, μ, U, and ε.

2. Dimensional Analysis of Variables (M, L, T System)

Variable	Symbol	SI Unit	Dimension (MLT)
Pressure loss per unit length	Δp/L	Pa/m	$ML^{-2}T^{-2}$
Pipe diameter	D	m	L
Fluid density	ρ	kg/m^3	ML^{-3}
Dynamic viscosity	μ	Pa·s	$ML^{-1}T^{-1}$
Mean flow velocity	U	m/s	LT^{-1}
Wall roughness	ε	m	L

3. Application of the Buckingham π Theorem

3.1 Algebraic Formulation

The pressure loss per unit length is expressed as a function of the other variables:

$$\frac{\Delta p}{L} = f(D, \rho, \mu, U, \varepsilon)$$

3.2 Number of Dimensionless Groups

There are n = 6 variables (Δp/L, D, ρ, μ, U, ε) and k = 3 fundamental dimensions (M, L, T). By the Buckingham π theorem, the number of independent dimensionless groups is:

$$p = n - k = 6 - 3 = 4 \;\; \rightarrow \;\; \pi_1, \pi_2, \pi_3, \pi_4$$

The repeating variables chosen are U, D, and ρ, since they collectively span all three fundamental dimensions and characterize the flow (U), geometry (D), and fluid properties (ρ).

3.3 Writing the π Groups with Unknown Exponents

Each π group is formed by multiplying the three repeating variables (with unknown exponents) by one of the remaining variables:

$$\pi_1 = U^{x_1} D^{y_1} \rho^{z_1} \cdot \Delta p = M^0 L^0 T^0$$

$$\pi_2 = U^{x_2} D^{y_2} \rho^{z_2} \cdot L = M^0 L^0 T^0$$

$$\pi_3 = U^{x_3} D^{y_3} \rho^{z_3} \cdot \varepsilon = M^0 L^0 T^0$$

$$\pi_4 = U^{x_4} D^{y_4} \rho^{z_4} \cdot \mu = M^0 L^0 T^0$$

3.4 Dimensional Equations for the Exponents

Replacing each quantity by its dimensions and equating to $M^0L^0T^0$:

$$\pi_1 = \left(\frac{L}{T}\right)^{x_1} L^{y_1} \left(\frac{M}{L^3}\right)^{z_1} \left(\frac{M}{LT^2}\right) = M^0 L^0 T^0$$

$$\pi_2 = \left(\frac{L}{T}\right)^{x_2} L^{y_2} \left(\frac{M}{L^3}\right)^{z_2} \cdot L = M^0 L^0 T^0$$

$$\pi_3 = \left(\frac{L}{T}\right)^{x_3} L^{y_3} \left(\frac{M}{L^3}\right)^{z_3} \cdot L = M^0 L^0 T^0$$

$$\pi_4 = \left(\frac{L}{T}\right)^{x_4} L^{y_4} \left(\frac{M}{L^3}\right)^{z_4} \left(\frac{M}{LT}\right) = M^0 L^0 T^0$$

3.5 Solving for Exponents — Results for Each π Group

- ✓ For π_1 (pressure loss term):

$$\pi_1 = M^{(z_1+1)} L^{(x_1+y_1-3z_1-1)} T^{(-x_1-2)} = M^0 L^0 T^0$$

Equating exponents of M, L, T separately: $z_1 = -1$, $y_1 = 0$, $x_1 = -2$. Therefore:

$$\pi_1 = \frac{\Delta p}{\rho U^2}$$

✓ For π_2 (pipe length ratio):

$$\pi_2 = M^{(z_2)} L^{(x_2+y_2-3z_2+1)} T^{(-x_2)} = M^0 L^0 T^0$$

Equating exponents: $z_2 = 0$, $y_2 = -1$, $x_2 = 0$. Therefore:

$$\pi_2 = \frac{L}{D}$$

✓ For π_3 (relative roughness):

Since ε has the same dimension as L, by analogy with π_2: $x_3 = 0$, $y_3 = -1$, $z_3 = 0$. Therefore:

$$\pi_3 = \frac{\varepsilon}{D}$$

✓ For π_4 (inverse Reynolds number):

Solving: $x_4 = -1$, $y_4 = -1$, $z_4 = -1$. Therefore:

$$\pi_4 = \frac{\mu}{\rho U D} = \frac{1}{Re}$$

4. Final Dimensionless Relation

The general functional relation between the π groups is:

$$\pi_1 = f(\pi_2, \pi_3, \pi_4)$$

Substituting the expressions found for each π group, the final result is:

$$\frac{\Delta p}{\rho U^2} = f\left(\frac{L}{D}, \frac{\varepsilon}{D}, \frac{1}{Re}\right)$$

This result shows that the dimensionless pressure loss $\Delta p/(\rho U^2)$ depends on three dimensionless parameters: the slenderness ratio L/D, the relative roughness ε/D, and the inverse of the Reynolds number $1/Re = \mu/(\rho UD)$. This relation forms the theoretical basis of the Darcy-Weisbach equation and the Moody diagram — two fundamental tools in pipe flow analysis and hydraulic engineering.

Exercise N°5:

Oil with a relative density of 0.860 is pumped through a horizontal tube of diameter D = 5 cm and length L = 300 m. The volumetric flow rate is Q_V = 1.20 l/s. The flow is assumed to be laminar. The linear head loss across this pipe section is measured to be 21 mCE (meters of water column).

Determine the kinematic and dynamic viscosities of the oil and find the flow Reynolds number.

Solution N°5:

1. Velocity of the Flow (V):

$$V = \frac{Q_V}{A} = 4 \cdot \frac{Q_V}{\pi D2} = 4 \times \frac{1.2 \times 10 - 3}{\pi \times (0.05)2} = 0.611\, m/s$$

2. Friction Loss Coefficient (λ):

The linear head loss expressed in terms of water column (mCE) introduces a pressure differential:

$$\Delta p = \rho_{H2O}.g.\Delta h$$

Relating this to Darcy-Weisbach's equation for the oil fluid column:

$$\lambda = \frac{\rho H2O \cdot \Delta h \cdot D \cdot 2g}{\rho oil \cdot L \cdot V2} = \frac{1 \times 21 \times 0.05 \times 2 \times 9.81}{0.860 \times 300 \times 0.6112} = 0.214$$

3. Kinematic Viscosity (ν) and Dynamic Viscosity (μ):

Using Poiseuille's law for fully developed laminar fluid flow (λ = 64/Re):

$$\nu = \frac{\lambda \cdot V \cdot D}{64} = \frac{0.214 \times 0.611 \times 0.05}{64} = 1.02 \times 10 - 4\, m^2/s$$

Dynamic viscosity is evaluated as:

$$\mu = \rho \cdot \nu = (0.860 \times 1000) \times 1.02 \times 10 - 4 = 0.0877\, Pa \cdot s$$

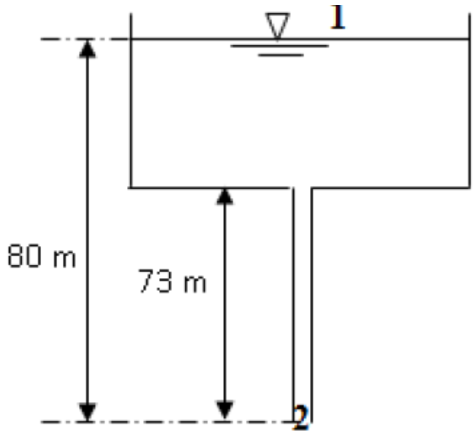
4. Verification of Laminar Regime:

$$Re = \frac{64}{\lambda} = \frac{64}{0.214} = 299$$

Since Re = 299 << 2300, the assumption of a stable laminar flow regime is validated.

Exercise N°6:

A water reservoir is drained by a vertical discharge pipe. The diameter of the pipe is 50 mm and its total length is 73 m. Neglecting minor localized losses and assuming a linear friction factor coefficient λ = 0.025, calculate the steady-state volumetric discharge flow rate.



Solution N°6:

Applying the extended Bernoulli energy equation between the free water surface (1) and the final discharge exit point (2):

$$\frac{P1}{\rho g} + \frac{V1}{2g} + Z1 = \frac{P2}{\rho g} + \frac{V2}{2g} + Z2 + \Delta H$$

Given boundary conditions from the schematic:

- ✓ $Z_1 = 80$ m, $Z_2 = 0$ m (datum at exit line)
- ✓ $P_1 = P_2 = P\_atm$ (both boundaries are vented to atmosphere)
- ✓ $V_1 \approx 0$ m/s (large reservoir assumption)

Substituting the friction head loss expression ($\Delta H = \lambda \cdot L/D \cdot V_2^2 / 2g$) gives:

$$\frac{V2}{2g} + \lambda \cdot \frac{L}{D} \cdot \frac{V2}{2g} = 80$$

Isolating and calculating the fluid exit velocity V_2:

$$V2 = \sqrt{\frac{2g \cdot 80}{1 + \lambda \cdot \frac{L}{D}}} = \sqrt{\frac{2 \times 9.81 \times 80}{1 + 0.025 \times \frac{73}{0.05}}} = 6.47\ m/s$$

The volumetric flow rate Q is derived from the cross-sectional discharge area S_2:

$$Q = V2 \cdot S2 = 6.47 \times \frac{3.14 \times 0.052}{4} = 0.0127\ m3/s$$

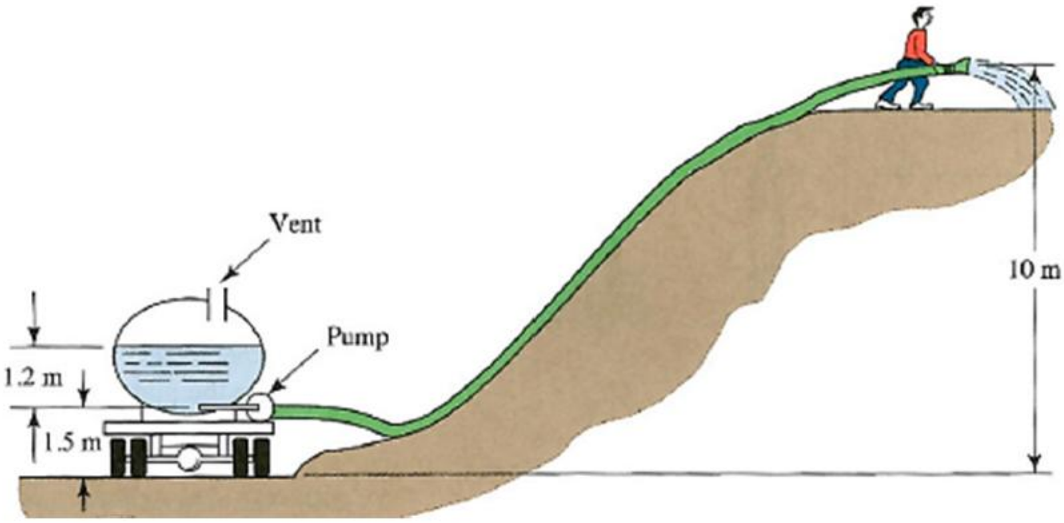
Exercise N°7

The figure below shows a liquid lawn fertilizer distribution system. The nozzle at the end of the pipe requires a minimum operating pressure of 140 kPa. The pipe is made of smooth plastic (ε = 0.03 mm) with an inner diameter of 25 mm. The fertilizer solution has a specific gravity of 1.10 and a dynamic viscosity of 2.0×10^{-3} Pa·s. The total pipe length is 85 m and the volume flow rate is 95 L/min.

The system geometry is as follows:

- ✓ Point (1): free surface of the fertilizer tank — elevation $z_1 = 1.5 + 1.2 = 2.7$ m
- ✓ Point (2): nozzle exit at the top of the hill — elevation $z_2 = 10$ m
- ✓ Point (3): pump outlet — elevation $z_3 = 1.5$ m

Neglect energy losses on the suction side of the pump (from tank to pump inlet).



Determine:

1. The power delivered by the pump to the fluid.
2. The pressure at the pump outlet (point 3).

Solution N°7:

1. Pump Power Delivered to the Fluid

Apply the Bernoulli Equation between Points (1) and (2)

Applying the extended Bernoulli equation from the tank free surface (1) to the nozzle exit (2), including pump head hp and friction head loss Δh_L:

$$\frac{p_1}{\rho g} + \frac{{V_1}^2}{2g} + z_1 + hP - \Delta h_L = \frac{p_2}{\rho g} + \frac{{V_2}^2}{2g} + z_2$$

Since the tank is open to atmosphere and its free surface is large: $p_1 = 0$ (gauge), $V_1 \approx 0$. Also, $V_2 = V_3 = V$ (same pipe diameter throughout). The Bernoulli equation simplifies to:

$$h_P = \frac{p_2}{\rho g} + \frac{{V_2}^2}{2g} + \Delta h_L + (z_2 - z_1)$$

$$V = \frac{Q_v}{A}$$

Qv = 95 L/min = Qv = 95/60 × 10^{-3} m^3/s. The cross-sectional area of the pipe is $A = \pi D^2/4$. The mean velocity is:

$$V = \frac{Q_v}{A} = \frac{\frac{95 \times 10^{-3}}{60}}{\frac{\pi (0.025)^2}{4}} = 3.23\ m/s$$

Determine the Flow Regime (Reynolds Number)

Using ρ = 1.10 × 1000 = 1100 kg/m^3:

$$Re = \frac{\rho V D}{\mu} = \frac{1100 \times 3.23 \times 0.025}{2.0 \times 10^{-3}} = 4.44 \times 10^4 \gg 3000$$

The flow is turbulent (Re >> 3000), so friction losses must be calculated using the Moody diagram.

Determine the Friction Factor from the Moody Diagram:

The relative roughness of the pipe is:

$$\frac{\varepsilon}{D} = \frac{0.0003}{0.025} = 0.012$$

From the Moody diagram, for Re = 4.44×10^4 and ε/D = 0.012, the Darcy friction factor is:

$$\lambda = 0.021$$

Calculate the Head Loss Due to Friction using the Darcy-Weisbach equation:

$$\Delta h_L = \lambda \frac{LV^2}{2gD} = 0.021 \times \frac{85 \times (3.23)^2}{2 \times 9.81 \times 0.025} = 37.96\ m$$

Calculate the Required Pump Head:

Substituting all values into the simplified Bernoulli equation: note that $z_2 - z_1$ =10−(1.5+1.2) =7.3 m:

$$h_P = \frac{140\,000}{1100 \times 9.81} + \frac{(3.23)^2}{2 \times 9.81} + 37.96 + (10 - (1.5 + 1.2)) = 58.7\ m$$

Calculate the Pump Power:

The hydraulic power delivered by the pump to the fluid is:

$$P_P = h_P \times \rho \times g \times Q_v$$

Qv = 95/60 × 10^{-3} = 1.58 × 10^{-3} m^3/s:

$$P_P = 58.7 \times 1100 \times 9.81 \times 1.58 \times 10^{-3} \approx 1000\ W$$

→ The pump delivers approximately 1 kW to the fluid.

2. Pressure at the Pump Outlet (Point 3)

Apply Bernoulli between Points (2) and (3)

Applying the Bernoulli equation from the pump outlet (3) to the nozzle (2). Since both cross-sections have the same diameter, $V_2 = V_3$, and the velocity terms cancel:

$$\frac{p_2}{\rho g} + \frac{{V_2}^2}{2g} + z_2 = \frac{p_3}{\rho g} + \frac{{V_3}^2}{2g} + z_3 - \Delta h_L$$

Since $V_2 = V_3$, the kinetic energy terms cancel. Rearranging for p_3:

$$p_3 = p_2 + \rho g((z_2 - z_3) + \Delta h_L)$$

Compute p_3 Numerically

With p_2 = 140 000 Pa, ρ = 1100 kg/m³, g = 9.81 m/s², z_2 = 10 m, z_3 = 1.5 m, and Δh_L= 37.96 m:

$$p_3 = 140\,000 + 1100 \times 9.81 \times [(10 - 1.5) + 37.96] = 641\,350\ Pa \approx 641.4\ kPa$$

→ The gauge pressure at the pump outlet is approximately 641.4 kPa.

Exercise N8:

Water flows from reservoir A to reservoir B under the effect of a gauge pressure P_0 applied on reservoir A. The flow passes through a pipe of diameter d = 300 mm, roughness ε = 0.3 mm, and length l = 170 m.

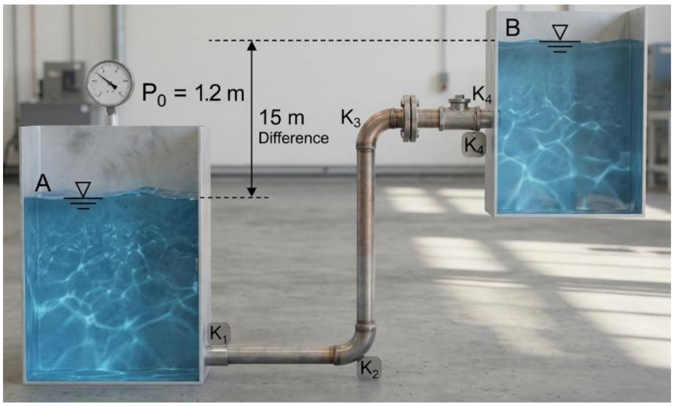
The singular head loss coefficients are:

- ✓ K_1 = 0.5 at the outlet of reservoir A,
- ✓ $K_2 = K_3$ = 0.15 for the two elbows,
- ✓ K_4 = 1 at the inlet of reservoir B.

Determine the gauge pressure P_0 required to obtain a volumetric flow rate Q_v = 200 L/s.

Given:

ρ = 10^3 kg/m³, g = 9.81 m/s², ν = 1.005 × 10^{-6} m²/s.)



Solution N°8:

Application of the Extended Bernoulli Equation between Reservoir A and Reservoir B:

$$\frac{PA}{\rho\, g} + \frac{V_A^2}{2g} + ZA = \frac{PB}{\rho g} + \frac{V_B^2}{2g} + ZB + \Delta H$$

Considering the following assumptions:

$ZA = 0$; $ZB = 15m$ and $PB = P_{atm}$ and $PA = P0$; $V_A = V_B = 0$

Bernoulli's equation becomes:

$$\frac{P0}{\rho g} = 15 + \lambda \frac{l}{d} . \frac{V^2}{2g} + (K1 + K2 + K3 + K4) \cdot \frac{V^2}{2g}$$

Where the total system head loss (ΔH) consists of both linear (major) and localized (minor) losses:

$$\Delta H = \Delta HL + \sum \Delta Hs = \lambda \cdot \frac{l}{d} \cdot \frac{V^2}{2g} + \sum Ki \cdot \frac{V^2}{2g}$$

The volumetric flow rate converted to SI units is Qv = 200 l/s = 0.2 m³/s. The velocity inside the pipe is:

$$V = \frac{Qv}{S} = \frac{4\, Qv}{\pi\, d^2} = \frac{4 \times 0{,}2}{\pi \times 0{,}3^2} = 2{,}83\, m/s$$

Calculation of Friction Factor (λ) using Reynolds Number and Relative Roughness: To extract λ from the Moody diagram, we calculate the dimensionless flow parameters:

• Relative Roughness:

$$\frac{\varepsilon}{d} = \frac{0{,}3}{300} = 0{,}001$$

• Reynolds Number (Re):

$$Re = \frac{d.V}{\nu} = \frac{0{,}3 \times 2{,}83}{1{,}005 \times 10^{-6}} = 844776 \ \ (Fully\ Turbulent\ Flow)$$

By reading the Moody diagram with a relative roughness of 0.001 and a Reynolds number of 844,776, the Darcy friction factor is evaluated at λ = 0.0199 (approximated as λ ≈ 0.02).

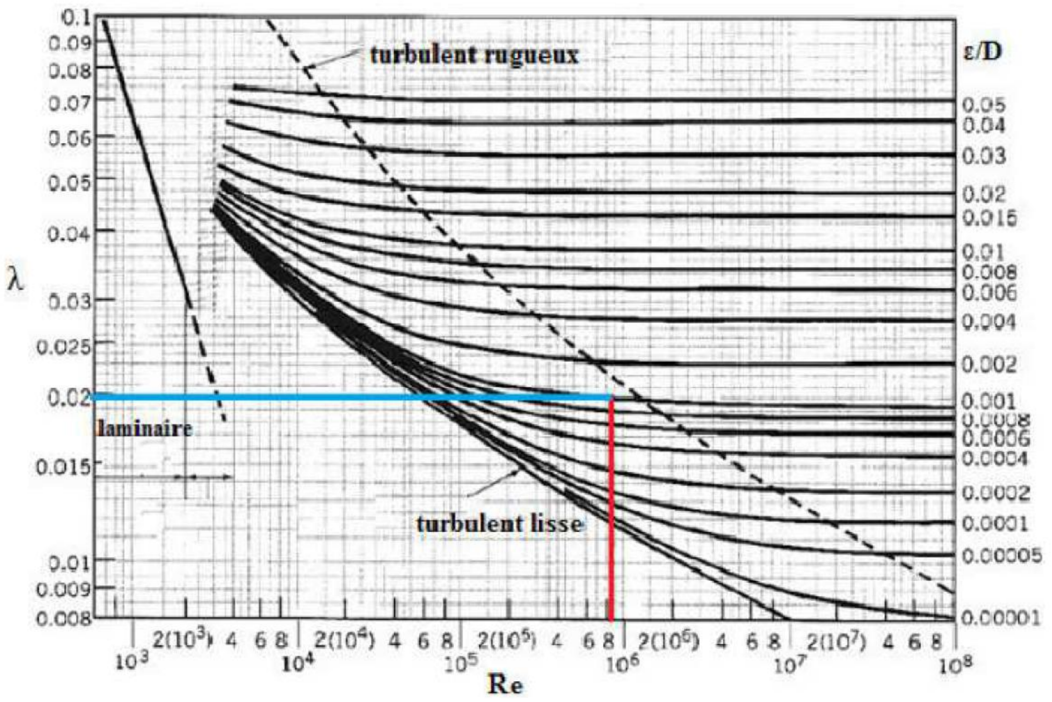
Substituting the evaluated flow parameters back into the simplified Bernoulli expression:

$$\frac{P0}{\rho\, g} = 15 + 0{,}02 \times \frac{170}{300} \times \frac{(2{,}83)^2}{2 \times 9{,}81} + (0{,}5 + 0{,}15 + 0{,}15 + 1) \times \frac{(2{,}83)^2}{2\ \times 9{,}81}$$

$$\frac{P0}{\rho\, g} = 19{,}99\, m$$

$$\Rightarrow P0 = 19{,}99 \times 103 \times 9{,}81 = 196102\, Pa\ = 196{,}1\, kPa$$

Final Answer: The required absolute gauge pressure P0= 196.1 kPa.

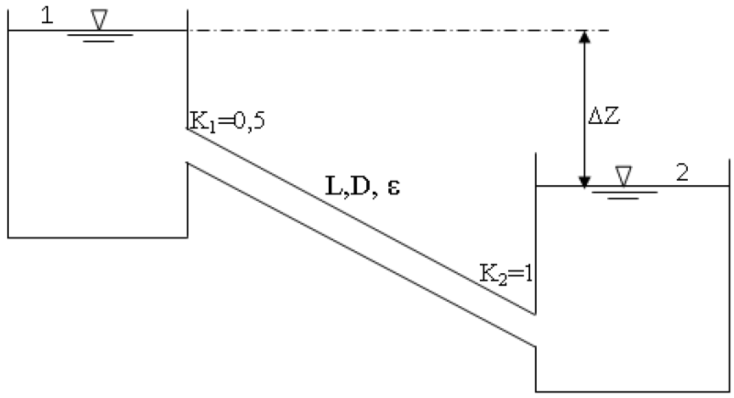


Exercise N°9:

Consider the system shown in the figure below, with the following data:

$\Delta Z = 45$ m: elevation difference between the two reservoir free surfaces, $L = 9000$ m: length of the riveted steel pipe ($\varepsilon = 0.9$ mm), $Q = 625$ l/s $= 0.625$ m^3/s: volumetric flow rate of water, $K_1 = 0.5$: minor loss coefficient at the exit of reservoir 1 (pipe entrance),$K_2 = 1.0$: minor loss coefficient at the entry of reservoir 2 (pipe exit).

Find the diameter D of the pipe using the Haaland equation to evaluate the Darcy–Weisbach friction factor λ.



Solution N°9:

Apply the Bernoulli Equation Between Surfaces 1 and 2

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + \Delta h_f$$

We have:

- ✓ $Z_1 = 0, Z_2 = \Delta Z = 45$ m
- ✓ $P_1 = P_2 = P_{atm}$ (both surfaces open to atmosphere)
- ✓ $V_1 = V_2 \approx 0$ (large reservoir surfaces, quasi-constant level)

The equation reduces to:

$$\Delta Z = \Delta h_f$$

The total head loss combines major (friction) losses and minor losses:

$$\Delta h_f = \lambda \frac{L}{D}\frac{V^2}{2g} + (K_1 + K_2)\frac{V^2}{2g}$$

$$\Delta h_f = \left(\lambda \frac{L}{D} + K_1 + K_2\right)\frac{V^2}{2g}$$

From the continuity equation $Q = \frac{\pi D^2}{4} V$ the velocity can be expressed as a function of the diameter:

$$V = \frac{4Q}{\pi D^2}$$

Substituting into the head loss equation:

$$\Delta Z = \left(\lambda \frac{L}{D} + K_1 + K_2\right)\frac{1}{2g}\left(\frac{4Q}{\pi D^2}\right)^2$$

Rearranging for D:

$$D = \left[\frac{8Q^2}{\pi^2 g\, \Delta Z}\left(\lambda \frac{L}{D} + K_1 + K_2\right)\right]^{1/5}$$

Since the friction factor λ depends on both the Reynolds number $Re = \frac{VD}{\nu}$ and the relative roughness ε/D, the diameter D appears on both sides of the equation. The solution is therefore obtained by iteration:

- ✓ Assume an initial diameter $D^{(0)}$
- ✓ Compute the mean velocity V
- ✓ Compute Re and ε/D
- ✓ Evaluate λ using the Haaland equation:

$$\frac{1}{\sqrt{\lambda}} = -1.8\log\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right]$$

- ✓ Compute a new diameter D
- ✓ Repeat until convergence $\left(\left|D^{(n+1)} - D^{(n)}\right| < \text{tolerance}\right)$

Fluid properties: water at 20 °C → $\nu = 10^{-6}\ \text{m}^2/\text{s}$

The following table summarizes the obtained results.

Iteration	$D^{(n)}$ (m)	V (m/s)	Re	ε/D	λ	D (m)
0	0.10	79.57	7.95×10^6	9.00×10^{-3}	0.0366	1.24
1	1.24	0.52	6.42×10^5	7.26×10^{-4}	0.0188	0.56
2	0.56	2.53	1.42×10^6	1.61×10^{-3}	0.0220	0.71
3	0.71	1.57	1.12×10^6	1.27×10^{-3}	0.0210	0.66

Iteration	$D^{(n)}$ (m)	V (m/s)	Re	ε/D	λ	D (m)
4	0.66	1.82	1.20×10^{6}	1.36×10^{-3}	0.0214	0.67
5	0.67	1.73	1.18×10^{6}	1.34×10^{-3}	0.0214	**0.67**

The iteration converges after 5 steps. The required pipe diameter is D=0.67 m.

Exercise N°10:

A water jet is supplied from a reservoir using a pump delivering a volumetric flow rate of qv=2 L/s. The flow passes through a pipe of length L=15 and diameter d=30 mm. The pipe also includes a 90° elbow with a minor head loss coefficient of Ks=0.3.

The free surface level of the reservoir, assumed to vary slowly with time, is located at an elevation of Z1=3 above the ground.

The water jet rises to a maximum height of Z2=10 m.

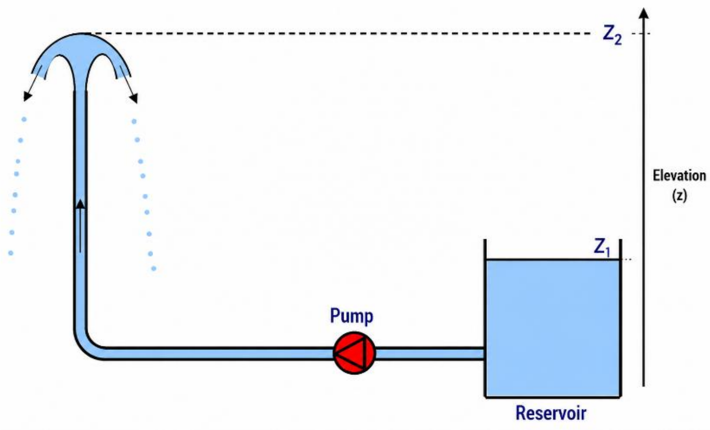
The following assumptions are considered:

The pressures at points 1 and 2 are atmospheric: P1=P2=Patm;

the dynamic viscosity of water is: μ=10−3 Pa.s ;

the density of water is: ρ=1000 kg/m^3 ;

5. Calculate the water flow velocity V in the pipe, expressed in m/s.
6. Determine the Reynolds number Re.
7. Specify the nature of the flow regime.
8. Determine the linear head loss coefficient λ, indicating the formula used.
9. Calculate the linear head losses ΔH in J/kg.
10. Calculate the minor head losses J minor in J/kg.
11. Apply Bernoulli's theorem between points (1) and (2) to determine the net pump power Pn in Watts.
12. Deduce the absorbed power Pa of the pump, knowing that its efficiency is η=75%.



Solution N° 10:

1. The mean flow velocity in the pipe is obtained from the continuity equation $q_v = A \cdot V$, where $A = \frac{\pi d^2}{4}$ is the pipe cross-sectional area:

$$V = \frac{4\, q_v}{\pi\, d^2}$$

$$V = \frac{4 \times 2 \times 10^{-3}}{\pi \times (0.03)^2} = \frac{8 \times 10^{-3}}{2.827 \times 10^{-3}} \approx 2.83 \text{ m/s}$$

2. The Reynolds number characterises the flow regime. It is defined as:

$$Re = \frac{\rho\, V\, d}{\mu} = \frac{V\, d}{\nu}$$

$$Re = \frac{2.83 \times 0.03}{\frac{10^{-3}}{10^{3}}} = \frac{2.83 \times 0.03}{10^{-6}} = 84900$$

3. Since $2000 < Re = 84\,900 < 10^5$, the flow is in the turbulent smooth regime.
4. Linear Friction Factor λ

Since the flow is turbulent smooth ($Re < 10^5$), the Blasius formula applies:

$$\lambda = 0.316 \times Re^{-0.25}$$

$$\lambda = 0.316 \times (84\,900)^{-0.25} = 0.316 \times 0.0570 \approx 0.018$$

5. Linear Head Losses

Linear (or major) head losses are given by the Darcy–Weisbach equation. In energy per unit mass (J/kg):

$$\Delta H_{linear} = -\lambda \cdot \frac{V^2}{2} \cdot \frac{L}{d}$$

The negative sign indicates an energy loss in the direction of flow.

$$\Delta H_{linear} = -0.018 \times \frac{(2.83)^2}{2} \times \frac{15}{0.03} = -0.018 \times 4.005 \times 500 \approx -37 \text{ J/kg}$$

6. Minor Head Losses

Minor (or singular) head losses due to the 90° elbow are expressed as:

$$\Delta H_{minor} = -K_s \cdot \frac{V^2}{2}$$

$$\Delta H_{minor} = -0.3 \times \frac{(2.83)^2}{2} = -0.3 \times 4.005 \approx -1.2 \text{ J/kg}$$

7. The generalised Bernoulli equation between point (1) (reservoir free surface) and point (2) (jet exit), including the pump energy input and head losses, is:

$$\frac{1}{2}(V_2^2 - V_1^2) + \frac{1}{\rho}(P_2 - P_1) + g(Z_2 - Z_1) = \frac{P_n}{\rho\, q_v} + \Delta H_{linear} + \Delta H_{minor}$$

- ✓ $V_1 = V_2$ (same pipe diameter, same flow rate, continuity)
- ✓ $P_1 = P_2 = P_{atm}$ (both ends open to atmosphere)

The equation reduces to:

$$g(Z_2 - Z_1) = \frac{P_n}{\rho\, q_v} + \Delta H_{linear} + \Delta H_{minor}$$

Rearranging for the net pump power:

$$P_n = \rho\, q_v[g(Z_2 - Z_1) - (J_{linear} + J_{minor})]$$

$$P_n = 1000 \times 2 \times 10^{-3} \times [9.81 \times (10 - 3) - (-37 - 1.2)]$$

$$P_n = 2 \times [9.81 \times 7 + 38.2] = 2 \times [68.67 + 38.2] = 2 \times 106.87$$

$$\boxed{P_n \approx 213.74 \text{ W}}$$

8. The absorbed (shaft) power P_a accounts for mechanical losses inside the pump through the efficiency η:

$$P_a = \frac{P_n}{\eta}$$

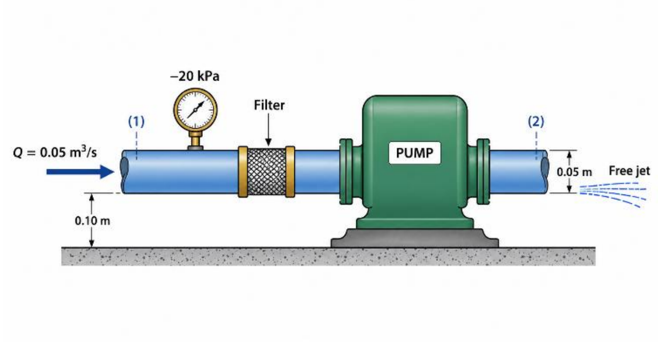
$$P_a = \frac{213.74}{0.75} \approx 284.99 \text{ W}$$

Exercise N°11:

The pump shown in the figure adds 20 kW of power to the flowing water. The only head loss in the system is the one occurring through the filter located at the pump inlet. Determine the head loss caused by this filter.

Given parameters from the diagram:

- ✓ Volumetric flow rate Q_v= 0.05 m³/s
- ✓ Inlet section (1): pressure p_1= -20 kPa, diameter d1 = 0.1 m
- ✓ Outlet section (2): pressure p_2 = 0 (Free Jet), diameter d2 = 0.05 m



Solution N°11:

Let us write the generalized Bernoulli equation for this flow:

$$\frac{p_1}{\rho g} + \frac{{V_1}^2}{2g} + z_1 + h_P = \frac{p_2}{\rho g} + \frac{{V_2}^2}{2g} + z_2 + \Delta h_L \quad (1)$$

Where, the pipeline centerline is horizontal, meaning the elevation heads are equal:

$$z_1 = z_2$$

The pressures are:

$$p_1 = -20\, kPa\, ;\, p_2 = 0$$

And, fluid velocities are calculated from the flow rate and cross-sectional areas:

$$V_1 = \frac{Q_v}{A_1}$$

$$V_2 = \frac{Q_v}{A_2}$$

Therefore, substituting the given values:

$$V_1 = \frac{Q_v}{A_1} = \frac{0,05}{\frac{\pi(0,1)^2}{4}} = 6,37\ m/s$$

$$V_2 = \frac{Q_v}{A_2} = \frac{0,05}{\frac{\pi(0,05)^2}{4}} = 25,5\ m/s$$

Also, calculating the head added by the pump (h_P):

$$h_P = \frac{\dot{W}}{\rho g Q_v} = \frac{20000}{10^3 \cdot 9,81 \cdot 0,05} = 40,8\ m$$

Then, equation (1) becomes:

$$\frac{-20000}{10^3 \cdot 9,81 \cdot 0,05} + \frac{6,37^2}{2 \cdot 9,81} + 40,8 = \frac{25,5^2}{2 \cdot 9,81} + \Delta h_L$$

Which yields:

$$\Delta h_L = 7,69\ m.$$

Annexe A. Dimensions and units

Any physical quantity can be characterized by dimensions. The magnitudes assigned to the dimensions are called units. Some basic dimensions such as mass *m*, length *L*, time *t*, and temperature *T* are selected as primary or fundamental dimensions, while others such as velocity *V*, energy *E*, and volume *V* are expressed in terms of the primary dimensions and are called secondary dimensions, or derived dimensions.

Several unit systems have been developed over the years. Two sets of units are still in common use today: the English system, which is also known as the United States Customary System (USCS), and the metric SI (from Système International d'Unités), which is also known as the International System.

Table 0-1: The seven fundamental (or primary) dimensions and their units in SI

Dimension	Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Temperature	kelvin (K)
Electric current	ampere (A)
Amount of light	candela (cd)
Amount of matter	mole (mol)

Table 0-2: Standard multiples

Multiple	Prefix	Multiple	Prefix
10^{24}	yotta, Y		yotta, Y
10^{21}	zetta, Z	10^{-1}	deci, d
10^{18}	exa, E	10^{2}	centi, c
10^{15}	peta, P	10^{3}	milli, m
10^{12}	tera, T	10^{-6}	micro, µ
10^{9}	giga, G	10^{-9}	nano, n
10^{6}	mega, M	10^{-12}	pico, p
10^{3}	kilo, k	10^{-15}	femto, f
10^{2}	hecto, h	10^{-18}	atto, a
10^{1}	deka, da	10^{-21}	zepto, z

Bibliography

S. Amiroudiene, J-L. Battaglia, *Mécanique des fluides*, 2ème édition, Dunod, Paris, 2011.

I. Paraschivoiu, M. Prud'homme, L. Robillard, *Mécanique des fluides*, Presses inter Polytechnique, Montréal, 2003.

M.S. Maalem, *Mécanique des fluides, Tome 1 : Hydrostatique, Hydrodynamique et Solutions binaires*, Sciences Exactes Physique, 1993.

B.R. Munson, T.H. Okiishi, W.W. Huebsch, A.P. Rothmayer, *Fundamentals of Fluid Mechanics*, 7th edition, John Wiley & Sons, Inc., Hoboken, New Jersey, 2013.

M. Potter, D.C. Wiggert, *Fluid Mechanics — Schaum's Outlines*, McGraw-Hill, New York, 2008.

A.I. Shalaby, *Fluid Mechanics for Civil and Environmental Engineers*, CRC Press, Taylor & Francis Group, Boca Raton, 2018.

R.V. Giles, J.B. Evett, C. Liu, *Fluid Mechanics and Hydraulics — Schaum's Outlines*, 4th edition, McGraw-Hill Education, New York, 2014.

J.B. Evett, C. Liu, *2500 Solved Problems in Fluid Mechanics & Hydraulics*, Revised First Edition, McGraw-Hill, New York, 1988.

J-B. Evett, R-V. Giles, C. Liu, *Mécanique des fluides et hydraulique — Cours et problèmes*, 2ème édition, Ediscience International, Paris, 1995.

R. Comolet, *Mécanique expérimentale des fluides, Vol. 1 : Statique et dynamique des fluides non visqueux*, 5ème édition, Elsevier Masson, Paris, 1997.

R. Comolet, *Mécanique expérimentale des fluides, Tome 2 : Dynamique des fluides réels — Turbomachines*, 4ème édition, Elsevier Masson, Paris, 1994.

J-P. Beaudry, J-C. Rolland, *Mécanique des fluides appliquée*, 2ème édition, Berger Eds, 2009.

W.H. Graf, M.S. Altinakar, *Hydrodynamique — Une Introduction*, Presses Polytechniques et Universitaires Romandes, Lausanne, 1995.